UNIFORM TILINGS OF THE HYPERBOLIC PLANE

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Abstract. A uniform tiling of the hyperbolic plane is a tessellation by regular geodesic polygons with the property that each vertex has the same vertex-type, which is a cyclic tuple of integers that determine the number of sides of the polygons surrounding the vertex. We determine combinatorial criteria for the existence, and uniqueness, of a uniform tiling with a given vertex type, and pose some open questions.

1. Introduction

A tiling of the hyperbolic plane is a partition into regular geodesic polygons (the tiles) which are non-overlapping (interiors are disjoint) and such that tiles which touch, do so either at exactly one vertex, or along exactly one edge. The vertex-type of a vertex $v$ is a cyclic tuple of integers $[k_1, k_2, \ldots, k_d]$ where $d$ is the degree (or valence) of $v$, and each $k_i$ (for $1 \leq i \leq d$) is the number of sides of the $i$-th of the $d$ polygons in counter-clockwise order around $v$. A uniform tiling is one in which the vertex-type is identical for each vertex (see Figure 1). A pair of tilings are identical if they are combinatorially isomorphic, that is, there is an orientation-preserving homeomorphism of the plane to itself that takes vertices, edges and tiles of one tiling to those of the other.

Uniform tilings of the Euclidean plane have been studied from antiquity, and it is known that there are exactly eleven such tilings, up to scaling (see [DM], and [GS77] for an informative survey). Also classical is the fact that there are 14 such uniform tilings of the round sphere, along with two infinite families (the prisms and antiprisms). Thirteen of these tilings have an automorphism group that is vertex-transitive, and are the famed Archimedean solids.

For a tiling of the hyperbolic plane, it is easy to verify that the vertex-type $k = [k_1, k_2, \ldots, k_d]$ of any vertex must satisfy

$$\alpha(k) = \sum_{i=1}^{d} \frac{k_i - 2}{k_i} > 2$$

(1)

since the sum of the interior angles of a regular hyperbolic polygon is strictly less than those of its Euclidean counterpart. We shall call $\alpha(k)$ the angle-sum of the cyclic tuple $k$.

It is well-known that there are infinitely many examples of uniform tilings with a vertex-type $k$ satisfying the above geometric condition. Indeed, the Fuchsian triangle groups $G(p, q)$ generated by reflections on the sides of a hyperbolic triangle with angles $\frac{\pi}{2}, \frac{\pi}{p}$ and $\frac{\pi}{q}$ generate a uniform tiling with vertex-type $[p^q] = [p, p, \ldots, p]$ whenever \( \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \) (see [EEK82]).

A basic question then is:
Question 1.1. Which cyclic tuples of integers \( k = [k_1, k_2, \ldots, k_d] \) satisfying \( \alpha(k) > 2 \) is the vertex-type for a uniform tiling of the hyperbolic plane? Which vertex-types have a unique such tiling?

Apart from the condition concerning the angle-sum, there is an immediate necessary combinatorial condition for the vertex-type of a uniform tiling, that arises from our requirement that the polygons corresponding to the integers in \( k \) appear in counter-clockwise order around each vertex. Namely,

(A) if an integer \( x \) follows an integer \( y \) in this cyclic tuple \([k_1, k_2, \ldots, k_d]\), that is, if \( xy \) appears, then so does \( yx \).

Indeed, if two polygons \( P_1 \) and \( P_2 \), with numbers of edges \( x \) and \( y \) respectively, share an edge, then they appear in different cyclic orders at the two endpoints of the common edge and contribute the two pairs \( xy \) and \( yx \) to the vertex-type (see Figure 2).

![Figure 2. Adjacent polygons with \( x \) and \( y \) sides appear in different orders around the endpoints of the common edge.](image)

The case when the degree \( d = 3 \), that is, when there are exactly three tiles around each vertex, is handled completely in Theorem 1.5 below. For the case when degree \( d \geq 4 \), we introduce another simple combinatorial condition:

(B) if \( xy \) and \( yz \) appear in the cyclic tuple \([k_1, k_2, \ldots, k_d]\), then so does \( xyz \).
We say \(uvw\) “appears” in a cyclic tuple \(k\) if the integers \(u, v, w\) appear in consecutive order in \(k\), and similarly for tuples of other lengths.

We prove the following sufficient criterion for the existence of “triangle-free” tilings:

**Theorem 1.2** (Existence criteria - I). Consider a cyclic tuple \(k = [k_1, k_2, ..., k_d]\) such that

- the angle-sum \(\alpha(k) > 2\),
- conditions (A) and (B) are satisfied,
- \(d \geq 4\), and each \(k_i \geq 4\).

Then there exists a uniform tiling of the hyperbolic plane with vertex-type \(k\). Moreover, this tiling is unique (up to an isomorphism) if any two consecutive elements of the cyclic tuple uniquely determine the rest of the tuple.

**Remark.** It is a folklore result that for a general vertex-type, uniqueness does not hold. Indeed, in §4, we shall show that there are uncountably many pairwise distinct uniform tilings of the hyperbolic plane with the same vertex type \(k = [4, 4, 4, 6]\). Note that such a cyclic tuple does not satisfy the criterion for uniqueness.

To handle the case of tilings with triangular tiles, we need to consider degree \(d \geq 6\), and an additional combinatorial condition:

(C) if the triples \(x3y\) and \(3yz\) appear in the cyclic \(k = [k_1, k_2, ..., k_d]\), then so does the 4-tuple \(x3yz\).

We shall prove:

**Theorem 1.3** (Existence criteria - II). Consider a cyclic tuple \(k = [k_1, k_2, ..., k_d]\) such that

- the angle-sum \(\alpha(k) > 2\),
- conditions (A), (B) and (C) are satisfied, and
- \(d \geq 6\).

Then there exists a uniform tiling of the hyperbolic plane with vertex-type \(k\). Moreover, this tiling is unique (up to an isomorphism) if any two consecutive elements of the cyclic tuple uniquely determine the rest of the tuple.

As a corollary, we obtain (see the end of §3) the uniqueness of the uniform tilings generated by the Fuchsian triangle-groups mentioned above:

**Corollary 1.4.** A uniform tiling of the hyperbolic plane with vertex type \([p^q]\) (where \(\frac{1}{p} + \frac{1}{q} < \frac{1}{2}\)) is unique, that is, any pair of such tilings are related by an orientation-preserving isometry of the hyperbolic plane, that takes vertices and edges of one to those of the other.

In §5, we provide the following necessary and sufficient conditions for the existence of uniform tilings with degree \(d = 3\):

**Theorem 1.5.** A cyclic tuple \(k = [k_1, k_2, k_3]\) is the vertex-type of a uniform tiling of the hyperbolic plane if and only if one of the following hold:

- \(k = [p, p, p]\) where \(p \geq 7\), or
- \(k = [2n, 2n, q]\) where \(2n \neq q\), and \(\frac{1}{n} + \frac{1}{q} < \frac{1}{2}\).

We now mention some questions that are still open.

First, our construction in Theorem 1.2 yields tilings that can have different symmetries, that is, could be invariant under different Fuchsian groups (or none at all) some of which have compact quotients.
Thus, we can ask:

**Question 1.6.** Given a tuple \( k = [k_1, k_2, \ldots, k_d] \), which compact oriented hyperbolic surfaces have a uniform tiling with vertex-type \( k \)? How many such tilings does such a surface have?

**Remark.** For the vertex-type \( [p^q] \), the question above was answered in [EEK82], where it was shown that such a tiling exists whenever the appropriate Euler characteristic count holds. The case when the surface is a torus or Klein bottle, and the vertex-type is that of a Euclidean tiling, was dealt with in [DM17]. The work in [KN12] enumerates uniform tilings of surfaces of low genera which have a vertex-transitive automorphism group.

Second, the existence criterion (B) in Theorem 1.2 is not necessary – see, for example, Figure 3. It should be possible, albeit tedious, to enumerate all possible vertex-types for uniform tilings of degrees \( d = 4 \) or \( 5 \) (see [Mit]). For general degree \( d \), not all tuples \( k \) that satisfy the angle-sum condition (1) can be realized by a uniform tiling – an example of this, that can be checked easily, is the tuple \( k = [3, 3, 4, 4, 3, 3, 4, 4] \). Note that this \( k \) also does not satisfy condition (B).

However, a set of necessary and sufficient conditions akin to conditions (A), (B) and (C) seems elusive. Although the work in [Ren08] develops algorithms for some related problems, we do not know if the answer to the following question is known:

**Question 1.7.** Is there a finite-state automaton to test if a given tuple \( k \) is the vertex-type of a uniform tiling of the hyperbolic plane?

2. A tiling construction

In this section we prove Theorem 1.2 by describing a constructive procedure to tile the hyperbolic plane uniformly. As we shall see, our method shall work provided the cyclic tuple \( k \) satisfies the hypotheses of Theorem 1.2, including Conditions (A) and (B) mentioned in the introduction. Moreover, the algorithm will be free of choices under an additional hypothesis, proving the uniqueness statement in that case.

2.1. Initial step: a fan. We shall begin with region \( X_0 \) of the hyperbolic plane tiled by exactly \( d \) polygons of sides \( k_1, k_2, \ldots, k_d \) around an initial vertex \( V \). In what follows, we shall call such a configuration of tiles around a vertex a fan.

A standard continuity argument implies:
Lemma 2.1. There is a unique choice of a side-length $l_0 > 0$ for the polygons in $X_0$ such that the total angle around the vertex $V$ is exactly $2\pi$.

Proof. For sufficiently small $l > 0$, a regular hyperbolic polygon of $k_i$ sides and side length $l$ will be approximately Euclidean, and each interior angle will be close to $\pi(k_i - 2)/k_i$. This makes the total angle $\alpha(l)$ at vertex $V$ close to $\pi \sum_{i=1}^{d} \frac{k_i - 2}{k_i} > 2\pi$, since the vertex-type satisfies the angle-sum condition to be a hyperbolic tiling. On the other hand, for large $l \gg 0$, as the vertices of the regular polygons tend to the ideal boundary, each interior angle will be close to 0, since for any ideal polygon adjacent sides bound cusps. The total angle $\alpha(l)$ is then close to 0. In fact, elementary hyperbolic trigonometry shows that $\alpha$ is a strictly monotonic function of $l$. Hence, there is a unique intermediate value $l_0 \in \mathbb{R}^+$ for which $\alpha(l_0) = 2\pi$. □

Remarks. (i) Throughout, we shall use polygons with side-lengths equal to the $l_0$ obtained in previous lemma. Moreover, we shall represent a fan as a disk with an appropriate division into wedges, together with vertices added to resulting boundary arcs to add more sides – see Figure 4.

(ii) Question 1.1 is equivalent to asking:

Question 2.2. When can a hyperbolic fan be extended to a uniform tiling of the hyperbolic plane?

It is easy to see that the following two properties hold for the tiled region $X_0$:

**Property 1.** All boundary vertices have valence 2 or 3. Moreover, there is at least one boundary vertex of valence 2.

(Note that the second statement follows from the property that not all tiles are triangles, which is weaker than our assumption of a triangle-free tiling.)

**Property 2.** The tiled region is homeomorphic to a disk.

2.2. Inductive step. The tiling is constructed layer by layer, namely, we shall find a sequence of tiled regions

\[ X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots \]

such that their union is the entire hyperbolic plane, the interior vertex of each $X_i$ has vertex type $[k_1, k_2, \ldots, k_d]$. 
In the following construction, we shall describe how \( X_{i+1} \) is obtained from \( X_i \) by adding tiles around each boundary vertex of \( X_i \) (that is, completing a fan), such that each boundary vertex of \( X_i \) becomes an interior vertex of \( X_{i+1} \). Informally speaking, the tiles added to construct \( X_{i+1} \) form a "layer" around \( X_i \); the tiling of the hyperbolic plane is thus built by successively adding concentric layers.

We shall now describe the inductive step of the construction, namely, how to add tiles to expand from \( X_i \) to \( X_{i+1} \).

In the construction, we shall assume that Properties 1 and 2 (see §2.1) hold for \( X_i \). As we saw, these were true for \( i = 0 \), namely for the tiled region \( X_0 \), and shall verify it for \( X_{i+1} \) when we complete the construction.

As a consequence of Property 2, the boundary \( \partial X_i \) is a topological circle. Let the boundary vertices be \( v_0, v_1, \ldots, v_n \) in a counter-clockwise order. Note that the number \( n \) of vertices is certainly dependent on \( i \), and in fact grows exponentially with \( i \), but we shall suppress this dependence for the ease of notation.

Moreover, we shall choose this cyclic ordering such that \( v_0 \) is a vertex of valence 3, and \( v_n \) is a vertex of valence 2. (This is possible because Property 1 holds.)

**Completing a fan at \( v_0 \).** We begin by adding tiles to complete the fan \( F_0 \) at \( v_0 \).

![Figure 5](image)

**Figure 5.** The fan \( F_0 \) is completed at the boundary-vertex \( v_0 \) of \( X_i \). The added wedge is shown shaded.

Recall that \( v_0 \) has valence 3. Hence, \( v_0 \) is the common vertex of two polygons \( P \) and \( Q \) in \( X_i \). The topological operation of adding the fan can be viewed as follows: consider a semi-circular arc centered at \( v_0 \) in the exterior of \( X_i \) and with endpoints at \( v_1 \) and \( v_n \). We add \( d - 2 \) “spokes” to the resulting “wedge” containing \( v_0 \): this results in a fan around \( v_0 \) comprising the initial polygon \( P \), and exactly \( d - 2 \) triangles. Finally, we add more valence 2 vertices to the sub-arcs of the boundary of the wedge, if need be, in order to obtain \( d \) polygons with the desired number of sides and cyclic order prescribed by the vertex-type.

The fact that we can do so requires Condition (A): Namely, let \( e \) be the common edge between \( P \) and \( Q \), one of whose endpoints is \( v_0 \), and the other \( w \). (See Figure 5.) Note that if the number of sides of \( P \) and \( Q \) are \( x \) and \( y \) respectively, then \( yx \) appears in that order for vertex-type for the vertex \( w \). Condition (A) then ensures that \( xy \) appears in the cyclic
Figure 6. The fan $F_j$ added at $v_j$, that has valence $3$ after $F_{j-1}$ was added.

Completing a fan at $v_1, v_2, \ldots, v_{n-1}$. We then successively complete fans $F_j$ at $v_j$ for $1 \leq j \leq n-1$ as follows. Assume we have completed fans at $v_0, v_1, \ldots, v_{j-1}$. At the $j$-th stage, there are two cases:

Case I. The vertex $v_j$ has valence $3$. This implies that $v_j$ had valence $2$ in $X_i$; the additional edge incident to $v_j$ comes from the fan $F_{j-1}$ added at $v_{j-1}$. Let $P$ be the polygon in $X_i$ that has $v_j$ as a vertex, and let $Q$ be the polygon in the fan $F_{j-1}$ that has $v_j$ as a vertex. Note that the edge between $v_{j-1}$ and $v_j$ is the common edge of $P$ and $Q$. To describe the fan $F_j$ topologically, draw an arc in the exterior of $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{j-1}$, between $v_{j+1}$ and the vertex in $\partial F_{j-1}$ adjacent to $v_j$. (See Figure 6.) Divide this wedge region into $d - 3$ triangles by adding spokes, and as before, add an appropriate number of vertices to the resulting circular arcs to have polygons with more than three sides. Note that if the number of sides of $P$ and $Q$ are $x$ and $y$ respectively, then $yx$ appears in the vertex type of $v_{j-1}$, and Condition (A) ensures that $xy$ appears in the cyclic vector $[k_1, k_2, \ldots, k_d]$. Hence there is indeed a choice of such polygons that completes the fan $F_j$.

Case II. The vertex $v_j$ has valence $4$. Then $v_j$ has valence $3$ in $X_i$. Let $P$ and $Q$ be the polygons in $X_i$ sharing the vertex $v_j$, and having $x$ and $y$ sides respectively. Then $xy$ appears in the vertex type of $v_j$, and Condition (A) ensures that $xy$ appears in the cyclic vector $[k_1, k_2, \ldots, k_d]$. Hence there is indeed a choice of such polygons that completes the fan $F_j$.

Claim 1. The triple $xyz$ appears in the cyclic tuple $k = [k_1, k_2, \ldots, k_d]$.

The edge between $v_{j-1}$ and $v_j$ is common between $Q$ and $R$, and the vertex-type at $v_{j-1}$ includes $zy$. By condition (A), the tuple $k$ will have $yz$ also. We have already seen above
that $k$ contains $xy$. Hence, by condition (B), the triple $xyz$ appears in $k$. This proves Claim 1.

Hence, we can choose numbers of sides of successive polygons to follow $P, Q$ and $R$ around $v_j$ to complete a fan $F_j$. This we do by adding and subdividing a wedge, just as in Case I.

**Completing a fan at $v_n$.** Finally, we need to complete the final fan $F_n$ around $v_n$. Note that we had chosen $v_n$ to have valence 2 in $X_i$; after adding the fans $F_j$ for $0 \leq j \leq n-1$, $v_n$ has valence 4 in $X_i \cup \left( \bigcup_{j=0}^{n-1} F_j \right)$, where the two additional edges belong to $F_0$ and $F_{n-1}$. Consider the three polygons $Q, P$ and $R$ in counter-clockwise order around $v_n$, where $Q$ is a polygon in $F_0$, $P$ is a polygon in $X_i$, and $R$ is a polygon in $F_n$, each having $v_n$ as a vertex. Let $P, Q, R$ have $x, y, z$ sides respectively. (See Figure 8.)

**Claim 2.** The triple $yxz$ appears in the cyclic tuple $k = [k_1, k_2, \ldots, k_d]$.

Since $Q$ and $P$ share the edge between $v_n$ and $v_0$, and since they are successive polygons in counter-clockwise order in the fan of $v_0$, $xy$ appears in the vertex-type $[k_1, k_2, \ldots, k_d]$. Similarly, $R$ and $Q$ share the edge between $v_{n-1}$ and $v_n$, and are successive polygons in the fan of $v_{n-1}$, and so $zx$ appears in the vertex-type. Applying condition (A), both $yx$ and $xz$ appears in the vertex-type, and hence by condition (B), so does $yxz$. This completes the proof of Claim 2.

Hence by adding a wedge based at $v_n$ between the fans $F_0$ and $F_{n-1}$, and subdividing into polygons, there is a choice of numbers of sides such that the successive polygons $Q, P, R$ are completed to a fan $F_n$. Note that once again, we have implicitly used the assumption that $d \geq 4$, as in the process we are adding at least four polygons around $v_n$.

**Verifying Properties 1 and 2.** We can now define

$$X_{i+1} := X_i \cup F_0 \cup F_1 \cup \cdots \cup F_n,$$

where by construction, all the boundary vertices $v_0, v_1, \ldots, v_n$ are in the interior, and every interior vertex has vertex-type $k$.

By the inductive hypothesis, $X_i$ is topologically a disk, and by construction, at each step of adding a fan at a boundary vertex, one is adjoining a simply-connected wedge to an arc of the boundary. Hence $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_j$ is topologically a disk for each $j = 0, 1, 2, \ldots, n$, establishing Property 2 for $X_{i+1}$.
To check Property 1 for $X_{i+1}$, notice that the new boundary vertices are the vertices of the fans $F_0, F_1, \ldots, F_n$ that lie on the boundary arcs of the wedges that we added. Vertices that lie in the interior of such a boundary arc have valence 2 or 3, like for an isolated fan $X_0$. There must be one such vertex in the boundary of the wedge that is added, since there cannot be a triangular tile. Such a vertex will be of valence 2, verifying the second statement of Property 1.

Now a vertex $w$ that lies at the intersection of two adjacent wedges, say for $F_j$ and $F_{j-1}$, is the endpoint of the edge from $v_j$ to $w$ that is common to $F_j$ and $F_{j-1}$. (See Figure 7.) Note that there cannot be an edge from $w$ to $v_{j+1}$, since then $v_{j+1}wv_j$ will form a triangular tile, contradicting our assumption that our tiling is triangle-free. Similarly, there cannot be an edge from $w$ to $v_{j-1}$. Hence this boundary vertex $w$ is of valence 3.

Thus, all boundary vertices of $X_{i+1}$ have valence 2 or 3, verifying the first statement of Property 1 also.

2.3. The endgame. It only remains to show:

**Lemma 2.3.** The union of the tiled regions $X_i$ for $i \geq 0$ is the entire hyperbolic plane.

**Proof.** Clearly, the initial fan $X_0$ contains disk of hyperbolic radius $r_0 > 0$ centered at the central vertex $V$.

**Claim 3.** There is an $r > 0$ such that any point on the boundary of $X_{i+1}$ is at least a distance $r$ from $X_i$, for each $i \geq 0$.

It is enough to verify this for the portion $\alpha$ of the boundary of the fan $F_j$ that we completed at the vertex $v_j$, that is disjoint from the adjacent fans $F_{j-1}$ or $F_{j+1}$ (where the indices are taken modulo $n$). For a fixed vertex-type $k$, there are only finitely many configurations of hyperbolic polygons to check, and hence there is a minimum such distance of $\alpha$ from the vertex $v_j$. This proves Claim 3.

Thus, for each $i \geq 0$, the distance of the boundary of $X_i$ from $V$ is at least $r_0 + i \cdot r$, and hence the region $X_i$ includes a disk of hyperbolic radius $r_0 + i \cdot r$ centered at $V$. As $i \to \infty$, the radius tends to infinity, and hence we cover the entire hyperbolic plane. This proves the lemma.

This completes the proof of the existence statement of Theorem 1.2.

The uniqueness statement of Theorem 1.2 follows from the fact that if the vertex-type $k = [k_1, k_2, \ldots, k_d]$ has the stated property, then there is a unique way of completing the fan $F_j$ for each $0 \leq j \leq n$ in our construction. This is because at any such step, the partial fan that was already at $v_j$, had (at least) two polygons $P$ and $Q$ already in place around it. If $x$ and $y$ are the number of sides of $P$ and $Q$ respectively, then this determines two successive elements $x$ and $y$ in $k$. Then by the assumed property of $k$, there is a unique sequence of polygons that can follow $P$ and $Q$ in the final fan. This determines a unique way of choosing the subdivision of the added wedge to determine these polygons. Hence the $i$-th stage of the construction (expanding from $X_i$ to $X_{i+1}$) is determined uniquely for each $i \geq 0$, and thus the final tiling is determined uniquely.

3. Handling triangular tiles: Proof of Theorem 1.3

Suppose we now have a vertex-type $k = [k_1, k_2, \ldots, k_d]$ that satisfies the hypotheses of Theorem 1.3. This time, $d \geq 6$, but we could have $k_i = 3$ for some (or all) $i \in \{1, 2, \ldots, \}$.
Constructing the exhaustion. The construction is the inductive procedure same as before: we start with a fan $X_0$ around a single vertex $V$, and proceed to build a sequence of tiled regions

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots$$

which exhausts the hyperbolic plane, and such that each interior vertex of $X_i$ has vertex-type $k_i$ for each $i \geq 0$.

In what follows we shall point out some of the additional considerations that we handle in this new case, that differs from the proof of Theorem 1.2 in §3.

The key difference is this time a vertex of valence 4 may appear on the boundary of a tiled region $X_{i+1}$ after completing the fans for the boundary vertices of $X_i$. (See Figure 11.)

Each region $X_i$ shall satisfy Property 2 as before, but the following different analogue of Property 1:

**Property 1’.** The following properties hold for $X_i$:

(i) All boundary vertices in $\partial X_i$ have valence 2, 3 or 4 in $X_i$.

(ii) Any boundary vertex $v \in \partial X_i$ of valence 4 is the vertex of a triangular tile in $X_i$ that intersects the boundary $\partial X_i$ only at $v$.

(iii) Finally, either there is a boundary vertex of valence 2, or there is a boundary edge that belongs to a triangular tile.

It is easy to see that $X_0$ satisfies (i) and (ii) above. Also, (iii) is vacuously true as $\partial X_0$ does not have any vertex of valence 4. As mentioned, valence 4 vertices may arise on the boundary of $X_i$ for $i \geq 1$ because of the presence of triangular tiles.

As before, the boundary $\partial X_i$ is a topological circle because of Property 2, and we denote the boundary vertices of $\partial X_i$ by $v_0, v_1, \ldots, v_n$ in counter-clockwise order. We also require that:

(a) $v_0$ has valence 3 or valence 4.

(b) One of the two hold:
   
   - Either $v_0v_n$ is a boundary edge that belongs to a triangular tile, or
   - $v_n$ has valence 2 in $X_i$.

This is possible for $X_0$ since if $k = [3^d]$, then each boundary vertex has valence 3, and the first condition of (b) holds. Otherwise, as in §3, we can in fact choose $v_0$ to have valence 3 and $v_n$ to have valence 2, that is, satisfying the second condition of (b). For $X_i$ where $i \geq 1$, we shall verify that such a choice of $v_0$ and $v_n$ is possible at the end of the inductive step.

In what follows we shall complete fans $F_j$ around $v_j$ for each $0 \leq j \leq n$ as before, and define

$$X_{i+1} := X_i \cup F_0 \cup F_1 \cup \cdots \cup F_n$$

for each $i \geq 0$.

Completing the fan at $v_0$. When the valence of $v_0$ is 3, we complete the fan $F_0$ around $v_0$ exactly as in the construction for Theorem 1.2. When the valence of $v_0$ equals 4, then the three polygons of $X_i$ around $v_0$ have number of sides $x, 3$ and $y$ because of part (ii) of Property 1’. We need to ensure that we can continue placing polygons around $v_0$ to complete a fan, that is, we need to prove:

Claim 4. $x3y$ is a consecutive triple in the cyclic tuple $k = [k_1, k_2, \ldots, k_d]$. 

Let e and f be the two edges in the interior of $X_i$ that are incident on $v_0$, and let $w$ and $w'$ be their other endpoints. Note that part (iii) of Property 1' implies that e and f are in fact, two sides of a triangular tile. Suppose P and Q are the other polygons in $X_i$ with $v_0$ as a vertex, having $x$ and $y$ sides respectively. Then considering the vertices $w$ and $w'$ that lie in the interior of $X_i$ (and consequently have vertex-type $k$) we see that the pairs $3x$ and $3y$ must appear in the tuple $k$. By Condition (A), this implies that $x3$ and $3y$ appear in $k$, and by Condition (B), so does $x3y$. This completes the proof of Claim 4.

Then, as before, we can add a wedge at $v_0$ in the exterior of $X_i$, and subdivide into polygons by adding spokes and vertices on the resulting boundary arcs of the wedge, having the numbers of sides that determine the rest of the tuple $k$ following $x, 3$ and $y$. This completes the fan $F_0$ at $v_0$.

**Completing the fan at $v_j$.** Now suppose we have completed fans around $v_0, v_1, \ldots, v_{j-1}$ for $1 \leq j \leq n - 1$, and we need to complete the fan $F_j$ at $v_j$.

As before, our analysis divides into cases depending on the valency of the vertex $v_j$ in $X_i$. When $v_j$ has valence 3 or 4 in the already-tiled region $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{j-1}$, the completion of the fan $F_j$ proceeds exactly as in the corresponding step in the proof of Theorem 1.2. The new case is when $v_j$ has valence 5, that is, when it had valence 4 in $X_i$ (before the other fans were completed). In this case, there are three polygons $P, T,$ and $Q$ in $X_i$ which shares a vertex $v_j$, where $T$ is a triangular tile, and there is another polygon $R$ in the fan $F_{j-1}$ that has $v_j$ as a vertex. (See Figure 9.)

**Figure 9.** Completing a fan $F_j$ at a vertex $v_j$ of valence 5.

Thus, around $v_j$, there are polygons $P, T, Q, R,$ in that counter-clockwise order. If the corresponding numbers of sides are $x, 3, y$ and $z$, in order to be able to complete a fan at $v_j$ with vertex-type $k$, we need to show:

**Claim 5.** The 4-tuple $x3yz$ appears as consecutive elements in $k$.

This is where we shall use Condition (C). Let $w$ and $w'$ be other endpoints of the triangular tile $T$ that has $v_j$ as a vertex. Note that $w, w'$ both have vertex-type $k$ by the inductive hypothesis, since they lie in the interior of $X_i$. Then, since the polygons $T$ and $P$ appear in counter-clockwise order around $w$, the pair $3x$ appears in $k$. Similarly, considering the polygons around $w'$, we see that the pair $y3$ appears in $k$. Using Conditions (A) and (B), we deduce, exactly as in a previous claim, that the triple $x3y$ appears in $k$. Now the vertex $v_{j-1}$ has the polygons $R$ and $Q$ in counter-clockwise order around it, so the pair $zy$ also appears in $k$. Applying the same argument involving Conditions (A) and (B), we conclude that $3yz$ appears in $k$. Finally, since the triples $x3y$ and $3yz$ are in $k$, an application of Condition (C) proves the claim. This completes the proof of Claim 5.
Figure 10. Completing the final fan in the case when \( v_0v_n \) is the edge of a triangular tile \( T \).

This allows a wedge to be added at \( v_j \), and divided into polygons, so that the polygons \( P, T, Q \) and \( R \) are part of a fan \( F_j \) of vertex-type \( k \) that is thus completed around \( v_j \).

**The final fan.** To complete the fan \( F_n \) at the remaining boundary-vertex \( v_n \), we would need to add a wedge that goes between the fans \( F_0 \) and \( F_{n-1} \), subdivide into polygons.

The case when \( v_n \) had valence 2 in \( X_i \) is exactly as in the case of completing the final fan in the proof of Theorem 1.2.

The remaining case is when \( v_0v_n \) is an edge of a triangular tile \( T \): in this case the polygons in \( X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{n-1} \) that share a vertex with \( v_n \) are \( P \) (which is part of \( F_0 \)), \( T \) and \( Q \) (which are part of \( X_i \)) and \( R \) (which is part of \( F_{n-1} \)), where \( P, T, Q \) and \( R \) are in counter-clockwise order around \( v_n \). (See Figure 10.) Suppose the numbers of vertices of \( P, Q \) and \( R \) are \( x, y \) and \( z \) respectively. Then, by exactly the same argument as in Claim 5, we have that \( xyz \) belongs to the vertex-type \( k \), and hence there is indeed a completion of these four polygons to a fan \( F_n \) at \( v_n \).

This completes the new tiled region \( X_{i+1} \).

**Verifying Property 1’ and completing the proof.** By construction, all interior vertices of \( X_{i+1} \) have vertex-type \( k \), and it is easy to see that Property 2 holds, namely, \( X_{i+1} \) is homeomorphic to a disk. Thus, it only remains to verify Property 1’, which is where the degree condition \( d \geq 6 \) is used.

The key observation is that when the fans \( F_j \) (for \( 0 \leq j \leq n \)) are added to \( X_i \), then the following holds:

**Claim 6.** A portion of the wedge added while completing \( F_j \) lies on the boundary of \( X_{i+1} \). In other words, \( F_j \setminus (F_{j-1} \cup F_{j+1}) \) is not empty.

For example, when a fan \( F_j \) is added to a boundary vertex \( v_j \in \partial X_i \) having valence 4 in \( X_i \), then there are already four polygons around \( v_j \) in \( X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{j-1} \), and the added wedge (to complete the fan \( F_j \)) needs to have at least one spoke, since the total number of polygons need to be at least 6. If \( q \) is the endpoint of the first spoke (in counter-clockwise order around \( v_j \)), then \( F_{j+1} \cap F_j \) cannot contain the portion of the wedge boundary that lies between \( q \) and \( F_{j-1} \). Hence this portion of the boundary of \( F_j \) is on the boundary of \( X_{i+1} \).
The same holds for the other cases (when \( v_j \) has valencies 2 or 3); note that then the added wedge needs to be divided with even more spokes, to have a final valence at least 6. This proves Claim 6.

Recall now that Property 1’ had three parts.

Proof of (i) and (ii). A valence 4 vertex is created in the boundary of \( X_{i+1} \) when the fan \( F_j \) (for \( 0 \leq j \leq n - 1 \)) has a triangular tile \( T \) in the subdivided wedge, one of whose edges is \( v_j v_{j+1} \). In that case, if \( q \) is the other vertex of \( T \), then \( q \) lies in the boundary of \( X_{i+1} \), and is disjoint from \( F_{j-1} \), by the claim above. Moreover, it has valence 4 on \( X_{i+1} \), since the edge \( qv_{j+1} \) will be shared by a polygon in the wedge added at \( v_{j+1} \) to complete the next fan \( F_{j+1} \). (See Figure 11.) The only other case when a valence 4 vertex appears in the boundary of \( X_{i+1} \) is when the fan \( F_0 \) has a triangular tile in the added wedge that has side \( v_0 v_n \). Then, the other vertex \( q \) of \( T \) lies in the boundary of \( X_{i+1} \), and has valence 4 in \( X_{i+1} \) when the wedge (for \( F_n \)) is added at \( v_n \). In both these cases, the triangular tile \( T \) lies in the interior of \( X_{i+1} \), and (ii) is satisfied.

In all other cases, when completing a fan, the extreme points of the boundary of any added wedge has valence 3. Note that if a portion \( \alpha \) of the boundary of an added wedge lies in the boundary of \( X_{i+1} \), then \( \alpha \) is also a portion of the boundary of a fan, with no other edges from it to other parts of \( X_{i+1} \), and hence all vertices that lie in \( \alpha \) have valence either 2 or 3. This proves (i).

Proof of (iii). Finally, recall that the subdivision of the wedge into polygons involves adding spokes, and then, in the case of non-triangular tiles, adding valence 2 vertices to the resulting boundary arcs to achieve the desired number of sides. We claim above implies that there is a portion \( \alpha \) of an added wedge that lies in the boundary of \( X_{i+1} \). This boundary arc \( \alpha \) is either the boundary of (one or more) triangular tiles tiling the wedge, or there is some polygon in the added wedge having a number of sides greater than 3. In the latter case, the subdivision procedure implies that there is a vertex in \( \alpha \), and consequently in the boundary of \( X_{i+1} \), that has valence 2. In the former case, there is a boundary edge of \( X_{i+1} \) that belongs to a triangular tile. This proves (iii).

Thus \( X_{i+1} \) satisfies Property 1’, and this completes the inductive step.

Finally, we verify that we can choose an ordering of the new boundary vertices \( v'_0, v'_1, \ldots, v'_n \) of \( X_{i+1} \) such that \( v'_0 \) and \( v'_n \) satisfy (a) and (b) stated after Property 1’. In fact, these successive vertices \( v'_n, v'_0 \) can be chosen to lie along the boundary of the final wedge added while
completing $F_\alpha$. Indeed, an extreme point $w$ (in counter-clockwise order) in the boundary of such a wedge, that also belongs to $F_\alpha$, has valence 3 or 4. This satisfies (a). The vertex $v$ in the wedge boundary that precedes $w$ either has valence 2, in case the edge $wv$ belongs to a polygon in the wedge having more than three sides, or else $wv$ is an edge of a triangular tile in the added wedge. This satisfies (b), and thus, the vertices $w$ and $v$ can be taken to the first and last vertices ($v'_0$ and $v'_n$) respectively, in our new counter-clockwise ordering of the boundary vertices of $X_{i+1}$.

Thus, we get a sequence of nested tiled regions

$$X_0 \subset X_1 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots$$

such that any interior vertex of $X_i$ has vertex-type $k$. Lemma 2.3 still applies (its proof did not use the hypotheses of Theorem 1.2), and this sequence of regions exhausts the hyperbolic plane, defining the desired uniform tiling.

Proof of Corollary 1.4. We first show that two uniform tilings $T$ and $T'$ with the same vertex-type $k = [p^q]$ where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ are combinatorially isomorphic. Note that the above inequality arises from the angle-sum condition (1).

First, it is easy to see that the property of $k$ that implies uniqueness holds in the case $k = [p^q]$.

If $p = 3$, then the inequality arising from the angle-sum condition implies that $q \geq 7$. Hence, in this case, the hypotheses of Theorem 1.3 are satisfied by $k = [p^q]$, and we deduce that the two tilings are isomorphic.

If both $p, q \geq 4$ the the hypotheses of Theorem 1.2 are satisfied by $k$, and we similarly deduce that the two tilings are isomorphic.

Finally, if $q = 3$, then $p \geq 7$, and the tilings are the duals to uniform tilings with vertex-type $[3^p]$. The latter tilings are isomorphic, as noted above, and hence so are their duals.

Since $T$ and $T'$ are isomorphic, there is an orientation-preserving homeomorphism $h$ of the hyperbolic plane to itself, that maps vertices and edges of $T$ to those of $T'$. By Lemma 2.1 there is a unique choice of a hyperbolic length of the edges for a uniform tiling with vertex-type $k$. Thus, $h$ can be taken to be length-preserving on each edge, and this can be extended to be an isometry on each tile. Thus, we in fact have an orientation-preserving isometry of the hyperbolic plane to itself, that realizes the isomorphism between $T$ and $T'$.

□

4. Examples of non-uniqueness

In this section we give examples of distinct tilings with the same vertex-type.

Uncountably many distinct tilings. Consider the vertex type $k = [4, 4, 4, 6]$. In this case, there is a uniform tiling $T$ of vertex type $k$ such that there is an action of $\Gamma$, a free subgroup of $\text{PSL}_2(\mathbb{R})$ on 3 generators, that acts transitively on the hexagonal tiles. (See the tiling on the right in Figure 12.)

Notice that it has a $\Gamma$-invariant collection $\mathcal{R}$ of bi-infinite rows of squares $\{R_y | y \in \Gamma\}$ (see the rows of blue squares in Figure 12). Any such row $R_y$ has adjacent layers $L_+$ and $L_-$ that comprise alternating hexagons and squares.
Figure 12. Spot the difference: these are distinct uniform tilings with identical vertex-type \([4, 4, 4, 6]\).

Then, a tiling \(T'\) that is distinct from \(T\) is obtained by shifting one side of each \(R_{\gamma}\) relative to the other. For example, performing this shift for three such bi-infinite rows adjacent to alternating sides (left, right, and bottom) of the central red hexagon in \(T\) produces a new tiling (on the left in Figure 12).

The same technique works for the vertex type \([4, 4, 4, n]\) for \(n > 4\). There is a uniform tiling with this vertex-type which has an infinite collection \(\mathcal{R}\) of bi-infinite rows of squares. The relative shift as above can in fact be performed at any subset of the collection of rows \(\mathcal{R}\): for each row, the change in the tiling is shown below (Figure 13.) Since there are uncountably many such subsets of \(\mathcal{R}\), we obtain uncountably many distinct tilings.

Figure 13. A relative shift of the tiles on either of the row \(R_{\gamma}\) (shown shaded) produces a different tiling with the same vertex-type \([4, 4, 4, 5]\).

Other examples. Note our construction in §2 and §3 can be done starting with any initial tiled region \(X_0\) that satisfies Property 1 (in the case that no tile is triangular and \(d \geq 4\)) or Property 1′ (in the case \(d \geq 6\)) and Property 2, together with the property that each interior vertex has the same vertex-type \(k = [k_1, k_2, \ldots, k_d]\).

If \(k\) satisfies the hypotheses of Theorem 1.3 but has a pair \(xy\) of consecutive elements which can be completed to the same cyclic tuple in two different ways, and these choices show up while completing the fans, then the final uniform tilings could be different. (See Figure 14).

5. Degree 3 tilings

In this final section we give the following necessary and sufficient conditions for the existence of tilings of degree \(d = 3\):
Figure 14. The unshaded tiles form a fan around \( V \) with vertex-type \( k = [4,3,3,3,4,3] \). The figures show two ways of completing another fan with the same vertex-type at a boundary vertex \( u \). This choice, used judiciously in the tiling construction, gives rise to two distinct uniform tilings.

Proof of Theorem 1.5 Our proof divides into several cases which we handle separately.

Case 1: \( k = [p, p, p] \). Note that for the angle sum (1) to be satisfied, we have \( p \geq 7 \). Uniform tilings with this vertex-type \( k \) exist, as they are dual to the uniform tilings \([3^p]\) which exist by the Fuchsian triangle-group construction mentioned in the introduction.

Case 2: \( k = [p, q, r] \) where \( p, q, r \) are distinct. Such a triple does not satisfy our necessary condition (A), and hence a uniform tiling with vertex-type \( k \) cannot exist.

Case 3: \( k = [p, p, q] \) where \( p \neq q, \) and \( p \) is odd. In this case, suppose there is a uniform tiling with vertex-type \( k \). Consider a \( p \)-gon in this tiling, with vertices \( v_0, v_1, \ldots, v_{p-1} \), and edges \( e_i \) between \( v_i \) and \( v_{i+1} \) for \( 0 \leq i \leq p - 1 \) (considered modulo \( p \)). Since each of these vertices have degree 3, the edges alternately share an edge with a \( p \)-gon and \( q \)-gon, respectively. However, if \( p \) is odd, then there is a vertex with three \( p \)-gons or three \( q \)-gons around it, which contradicts uniformity. (See Figure 15.) Thus, there can be no uniform tiling with vertex-type \( k \). This argument also appears in [DM17], Lemma 2.2 (ii), in the context of maps on surfaces.

Case 4: \( k = [p, p, q] \) where \( p \neq q, \) and \( p \) is even. Let \( p = 2n \). Then the angle-sum in (1) can be easily seen to yield that \( \frac{1}{n} + \frac{1}{q} < \frac{1}{2} \). Note that this is exactly the same condition that implies
the existence of an \([n^q]\) tiling. We can in fact construct a uniform tiling with vertex-type \(k\) by modifying an \([n^q]\) tiling \(T_0\) in the following way: replace each vertex in \(T_0\) by a \(q\)-gon, which has vertices along the edges of \(T\). The tiles of \(T_0\) are now \(2n\)-gons, since each such tile acquires an extra edge from the \(q\)-gon added at each vertex, and there are \(n\) vertices. Although this construction is a *topological tiling*, we can replace them by regular hyperbolic polygons since the angle-sum condition is satisfied (c.f. Lemma 2.1). (See Figure 16.)

![Figure 16](image)

**Figure 16.** A uniform tiling with vertex type \(k = [12, 12, 4]\) (figure on the right) is obtained by introducing squares at each vertex of a uniform tiling with vertex-type \([6^4]\) (figure on the left).

This covers all possibilities for a triple \(k\), and completes the proof. \(\square\)

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**References**


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