

## Article

A Short Proof that Compact 2-Manifolds Can Be Triangulated.  
DOYLE, P.H.; MORAN, D.A.  
in: Inventiones mathematicae | Inventiones mathematicae - 5 |  
Periodical issue - 2  
3 Page(s) (160 - 162)



## Nutzungsbedingungen

DigiZeitschriften e.V. gewährt ein nicht exklusives, nicht übertragbares, persönliches und beschränktes Recht auf Nutzung dieses Dokuments. Dieses Dokument ist ausschließlich für den persönlichen, nicht kommerziellen Gebrauch bestimmt. Das Copyright bleibt bei den Herausgebern oder sonstigen Rechteinhabern. Als Nutzer sind Sie nicht dazu berechtigt, eine Lizenz zu übertragen, zu transferieren oder an Dritte weiter zu geben.

Die Nutzung stellt keine Übertragung des Eigentumsrechts an diesem Dokument dar und gilt vorbehaltlich der folgenden Einschränkungen:

Sie müssen auf sämtlichen Kopien dieses Dokuments alle Urheberrechtshinweise und sonstigen Hinweise auf gesetzlichen Schutz beibehalten; und Sie dürfen dieses Dokument nicht in irgend einer Weise abändern, noch dürfen Sie dieses Dokument für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, aufführen, vertreiben oder anderweitig nutzen; es sei denn, es liegt Ihnen eine schriftliche Genehmigung von DigiZeitschriften e.V. und vom Herausgeber oder sonstigen Rechteinhaber vor.

Mit dem Gebrauch von DigiZeitschriften e.V. und der Verwendung dieses Dokuments erkennen Sie die Nutzungsbedingungen an.

## Terms of use

DigiZeitschriften e.V. grants the non-exclusive, non-transferable, personal and restricted right of using this document. This document is intended for the personal, non-commercial use. The copyright belongs to the publisher or to other copyright holders. You do not have the right to transfer a licence or to give it to a third party.

Use does not represent a transfer of the copyright of this document, and the following restrictions apply:

You must abide by all notices of copyright or other legal protection for all copies taken from this document; and You may not change this document in any way, nor may you duplicate, exhibit, display, distribute or use this document for public or commercial reasons unless you have the written permission of DigiZeitschriften e.V. and the publisher or other copyright holders.

By using DigiZeitschriften e.V. and this document you agree to the conditions of use.

## Kontakt / Contact

[DigiZeitschriften e.V.](http://www.digizeitschriften.de)

Papendiek 14

37073 Goettingen

Email: [info@digizeitschriften.de](mailto:info@digizeitschriften.de)

## A Short Proof that Compact 2-Manifolds Can Be Triangulated

P. H. DOYLE and D. A. MORAN\* (East Lansing, Michigan)

The result mentioned in the title of this paper was first proved by RADO [1]; a proof can also be found in [2]. The idea for the present proof is that of the first-named author, who discovered it while investigating the possibilities of engulfing in low dimensions. It is shorter than previous proofs, and is presented in the interests of economy.

We commence by listing a few familiar facts from geometric topology:

*The Jordan-Schoenflies Theorem.* A simple closed curve  $J$  in  $E^2$  separates  $E^2$  into two regions. There exists a self-homeomorphism of  $E^2$  under which  $J$  is mapped onto a circle.

*Thickening an Arc.* Each arc in the interior of a 2-manifold lies in the interior of a 2-cell. This 2-cell can be chosen to be disjoint from any preassigned compact set in the complement of the arc.

*Cellularity and Quotients.* For our purposes, a cellular set  $K$  is one that can be written as the intersection of a sequence of 2-cells

$$K = \bigcap_{i=1}^{\infty} E_i, \quad \text{where } E_i \subset \text{Int } E_{i-1} \quad (i=2, 3, \dots).$$

If  $K$  is a cellular subset of a 2-manifold  $M$ , then  $M/K$  is homeomorphic to  $M$ . (The corresponding statement also holds in  $n$  dimensions; see [3], for example.)

We shall also have need for the following

**Lemma.** Let  $M$  be a closed 2-manifold, and let  $C$  be a connected subset of  $M$  which is the union of  $n$  simple closed curves,

$$C = \bigcup_{i=1}^n C_i.$$

Let  $A$  be a compact, totally-disconnected subset of  $C$ . Then  $A$  lies in the interior of a closed 2-cell in  $M$ . (A totally-disconnected set is characterized by the property that each of its connected subsets consists of at most one point.)

*Proof.* Since  $A$  is compact and totally-disconnected, some subarc  $S$  of  $C_1$  contains no points of  $A$ . Thicken  $C_1 - S$  to obtain a closed 2-cell  $D_1$  whose interior contains  $A \cap C_1$ .

\* Work on this paper was supported by National Science Foundation Grant GP-7126.

Next, suppose that a 2-cell  $D_k$  has been constructed to satisfy

$$A \cap \bigcup_{i=1}^k C_i \subset \text{Int } D_k.$$

$C_{k+1} - D_k$  is the union of a collection of countably many mutually disjoint open arcs  $X_\mu$  whose endpoints lie on  $\text{Bd } D_k$ . Each  $X_\mu$  contains some subarc  $S_\mu$  for which  $A \cap S_\mu = \emptyset$ , so that the two arcs comprising  $X_\mu - S_\mu$  can be thickened to form closed 2-cells  $E_{\mu 1}$  and  $E_{\mu 2}$  which (a) avoid each other, (b) avoid  $X_\lambda$  for  $\lambda \neq \mu$ , and (c) meet  $D_k$  in closed 2-cells. Then

$$D_{k+1} = D_k \cup \bigcup_{\mu} (E_{\mu 1} \cup E_{\mu 2})$$

is a closed 2-cell whose interior contains

$$A \cap \bigcup_{i=1}^{k+1} C_i. \quad \square$$

**Theorem.** Any closed 2-manifold  $M$  can be triangulated.

*Proof.* Cover  $M$  irreducibly by a finite collection of closed disks  $\{B_1, B_2, \dots, B_n\}$ , and put  $C_i = \text{Bd } B_i$ . Let  $A$  be the singular set of

$$C = \bigcup_{i=1}^n C_i,$$

*i.e.*  $A$  consists of those points of  $C$  which do not have 1-dimensional euclidean neighborhoods in  $C$ .  $A$  is compact and totally-disconnected, and, since the cover  $\{B_i\}$  was irreducible, no proper subset of  $\{B_i\}$  covers  $M$ ; thus  $C$  is connected, so by the lemma,  $A$  lies in the interior of a closed 2-cell  $D$  in  $M$ . Note that  $M - C$  is the union of a countable collection of mutually disjoint open 2-cells, each of whose closures in  $M$  is a closed 2-cell; this is an easy consequence of the Jordan-Schoenflies Theorem. Note also that  $C - D$  consists of countably many mutually disjoint open arcs whose endpoints all lie on  $\text{Bd } D$ .

Now  $D \subset U \subset M$ , where  $U$  is an open 2-cell, as can be seen by thickening  $\text{Bd } D$  slightly; hence  $D$  is cellular (again by the Jordan-Schoenflies Theorem), and so  $M_1 = M/D$  is topologically equivalent to  $M$ .  $M_1$  is the copy of  $M$  with which we shall work from this point on.

Let  $R \subset M_1$  be the image of  $\overline{C - D}$  under the quotient map; thus  $R$  is the one-point-union of a countable collection of simple closed curves, and is locally euclidean except at one point  $p$  (the image of  $D$  under the quotient map). Moreover,  $M_1 - R$  is topologically equivalent to  $M - (C \cup D)$ . Furthermore, any 2-cell neighborhood  $V$  of  $p$  will contain all but a finite number of the simple closed curves which comprise  $R$ ; this follows from the compactness of  $\overline{C - D}$ , hence of  $R$ . If each of the

simple closed curves lying within  $V$  is spanned by the disk it bounds, a cellular set  $T$ , consisting of a one-point-union of closed disks, is obtained. (To see that  $T$  is cellular, let  $N$  be any neighborhood of  $T$ . Then  $N$  contains a 2-cell neighborhood  $V'$  of  $p$ , which must in turn contain all but finitely many of the simple closed curves comprising  $R$ . Thickening the arcs of  $R$  which lie outside  $V'$  and appending the resulting cells to  $T \cup V'$  yields a cell  $Q$ , where  $T \subset \text{Int} Q \subset Q \subset N$ .) Therefore, passing to  $M_1/T$  if necessary, we lose no generality in assuming that  $R$  is an  $r$ -leafed rose whose complement in  $M_1$  is composed of finitely many components that are open 2-cells.

Finally, enclose  $p$  in a small closed 2-cell  $E$  meeting each simple closed curve in  $R$  in two points of its boundary,  $\text{Bd} E$ .  $E \cup R$  is then a disk with a finite number of mutually disjoint closed arcs  $A_1, A_2, \dots, A_r$  meeting  $\text{Bd} E$  in their endpoints. Each  $A_i$  may be enclosed (except for its end-points) in the interior of a closed disk meeting  $E$  in a pair of arcs on its boundary. By selecting these disks to be disjoint in pairs one obtains a triangulable 2-manifold  $N$  with boundary. The triangulation of  $N$  is now extended to the closure of each component of  $M_1 - N$ .  $\square$

**Corollary 1.** *Any compact 2-manifold can be triangulated.*

*Proof.* Include the boundary curves in the set  $C$  of the proof of the theorem; minor modifications of the above argument will convert it into a proof of the corollary.  $\square$

**Corollary 2.** *Any compact 2-manifold with non-void boundary is embeddable in  $E^3$ .*

*Proof.* Given such a manifold  $M$ ,  $M$  can be embedded in a closed manifold  $M_1$ . Let  $M_1 = E^2 \cup R$ , a standard decomposition [4], where  $R$  is an  $r$ -leafed rose disjoint from the open 2-cell  $E^2$ . Then a copy of  $M$  lies in every neighborhood of  $R$ , for any embedding of  $R$  in  $E^3$ .  $\square$

### References

1. RADÓ, T.: Über den Begriff der Riemannschen Fläche. *Acts. Litt. Sci. Szeged*, **2**, 101–121 (1925).
2. AHLFORS, L. V., and L. SARIO: *Riemann surfaces*. Princeton: Princeton University Press 1960.
3. DOUADY, A.: Plongements des sphères. *Séminaire Bourbaki* **13**, exposé 205 (1960).
4. DOYLE, P. H., and J. G. HOCKING: A decomposition theorem for  $n$ -dimensional manifolds. *Proc. Amer. Math. Soc.* **13**, 469–471 (1962).

Dr. P. H. DOYLE and Dr. D. A. MORAN  
Michigan State University  
Department of Mathematics  
East Lansing, Michigan 48823, USA

(Received December 8, 1967)