Math 232 Midterm 1

3:30-5pm September 21, 2017

1. State if the following statements are **true** or **false**, and justify your answer. **[20 points]**

   (a) Let S be a closed surface, and let p be any point on it. Then $\pi_1(S \setminus \{p\})$ is a free group.

   (b) If X is a simply-connected, then there is a point $x \in X$ such that $X \setminus \{x\}$ is also simply-connected.

   (c) The letter

   A

   is homotopy equivalent, but not homeomorphic, to the letter

   P

   when thought of as subsets of $\mathbb{R}^2$.

   (d) Let X be the space obtained by attaching a disk D^2 to a torus $S^1 \times S^1$ via an attaching map that identifies the boundary ∂D^2 with the subspace $S^1 \times \{x_0\}$ of the torus. Then $\pi_1(X) \approx \mathbb{Z}$.

   (e) Any group G is a subgroup of a free group.

2. (a) For a set $S \subset \mathbb{R}^N$, let

   $$\langle S \rangle = \left\{ \sum_{i=1}^{m} t_i u_i : u_1, u_2 \ldots u_m \in S, t_1, t_2 \ldots \in [0,1], \sum t_i = 1 \right\}$$

   that is, all finite convex combinations of elements of S.

   Show that $\langle S \rangle$ is the smallest convex set containing S and it is the intersection of all convex sets containing S. **[4 points]**

(b) Consider the 2-dimensional simplicial complex K on the vertex set $\{u_1 = (-1, 0, 0), u_2 = (1, 0, 0), v_1 = (0, -1, 0), v_2 = (0, 1, 0), w_1 = (0, 0, -1), w_2 = (0, 0, 1)\}$ whose maximal simplices are $\langle u_i, v_i, w_i \rangle$, for $1 \leq i, j, k \leq 2$.
Show that K triangulates the 2-sphere S^2. **[6 points]**

3. Let $X = S^1 \times D^2$.

(a) Show that the subspace $S^1 \times \partial D^2$ is not a retract of X.

(b) Show that the subspace $S^1 \times \{x_0\}$ where $x_0 \in \partial D^2$ is a deformation retract of X.

(c) Is the circle $\{x_1\} \times \partial D^2$ where $x_1 \in S^1$, a deformation retract of X? Justify your answer. **[4+3+3 points]**

4. Let $X_1 = S^1 \vee S^1$, and let a and b be the generators of $\pi_1(X_1)$. Consider the space X obtained by attaching a Möbius strip M and two disks D^2_α and D^2_β to X_1, where the attaching maps identify

- the boundary of M with the loop b in X_1,
- the boundary of D^2_α with the loop in X_1 described by the word b^2,
- the boundary of D^2_β with the loop in X_1 described by the word ab.

(a) Compute the fundamental group $\pi_1(X)$. **[6 points]**

(b) Describe a space obtained by identifying sides of some polygon in $\mathbb{R}^2$, that has the same fundamental group as X. **[4 points]**

(One description of a Möbius strip is: it is the space obtained by deleting a disk from $\mathbb{R}P^2$.)
