Math 232 Homework 2

Due on: September 5th 2017

1. Let A be a subspace of X, and let $a \in A$. In what follows let $i : A \to X$ be the inclusion map.

   (a) If A is a retract of X then show that the induced homomorphism $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective.

   (b) if A is a deformation retract of X then show that i is a homotopy equivalence.

   (c) Let $x \in S^1 \times S^1$. Show that $S^1 \times \{x\}$ is a retract of $S^1 \times S^1$, but not a deformation retract.

2. Show that the following are equivalent:

   (a) X is homotopy equivalent to a point.

   (b) Any point $x \in X$ is a deformation retract of X.

   (c) The identity map $id_X : X \to X$ is *null-homotopic*, that is, homotopic to a constant map.

   (These are all equivalent notions of "contractible").

3. Give an example of a space X, subspace A and two maps $f, g : X \to X$ such that:

   (a) Both f and g restrict to the identity map id_A on the subspace A.

   (b) f is homotopic to g.

   (c) There is **no** homotopy f_t between $f = f_0$ and $g = f_1$ that fixes A throughout, that is, such that $f_t|_A = id_A$ for all $t \in [0, 1]$.

4. Suppose $X = A \cup B$ where A and B are non-empty open sets in X, such that

   - A and B are both simply-connected.
   - $A \cap B$ is path-connected.

   Then show that X is simply-connected.

5. Prove that $\mathbb{R}^2 \backslash \{(0,0),(1,0)\}$ is not homeomorphic to $\mathbb{R}^3 \backslash \{(0,0,0),(1,0,0)\}$. *(Give a complete argument using only what has been taught so far.)*

6. Suppose $f : S^1 \to S^1$ is a map that satisfies $f(-x) = -f(x)$ for all $x \in S^1$.

   (a) Given **three** examples of such a map.

   (b) Show that f cannot be homotopic to a constant map.