SUGGESTED EXERCISES - 2

TEICHMÜLLER THEORY (MATH 191B), WINTER 2013-4

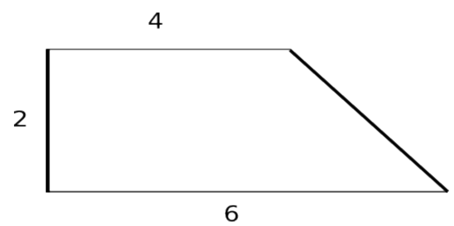
In what follows $\mathbb{D}$ is the interior of the unit disk, and $\lambda(\Gamma)$ denotes extremal length.

(1) Is there a quasiconformal homeomorphism $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \overline{\mathbb{D}}$?

(2) Let R be the rectangle $(0,4) \times (0,3) = \{(x,y) | 0 < x < 4, 0 < y < 3\}$ on the plane $\mathbb{R}^2 = \mathbb{C}$. Show that the map $f : R \to f(R)$ defined as $f(x,y) = (x+y,y)$ is quasiconformal. Compute its complex dilatation μ_f and the quasiconformal distortion $K(f)$. What is the image $f(R)$?

(3) Construct a quasiconfomal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $\mu_f \equiv \frac{1}{\sqrt{5}}$ in $\mathbb{D}$ and is zero elsewhere.

(4) Let Ω be the interior of the quadrilateral below (two horizontal sides and one vertical side have lengths as shown). Let Γ be the family of arcs between the sides shown in bold. Prove that $2 \leq \lambda(\Gamma) < \frac{30}{11}$.



(5) Let Ω be the square $(-1,1) \times (-1,1)$ and let R be the subset $(-\epsilon,\epsilon) \times (-\epsilon,\epsilon)$. Let Γ be the family of arcs between the vertical sides of Ω, that do not pass through R. Show that:

$$1 \leq \lambda(\Gamma) \leq \frac{1}{1-\epsilon}.$$

Hint: Adapt the lenth-area argument to this situation, to get the upper bound.

(6) Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal homeomorphism of the plane fixing the origin. Show that images of circles have uniformly bounded distortion, that is, there is a constant C depending only on K such that for any $r > 0$ we have:

$$\frac{\max\limits_{\theta \in [0,2\pi]} |f(re^{i\theta})|}{\min\limits_{\theta \in [0,2\pi]} |f(re^{i\theta})|} \leq C.$$

(7) Let $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a homeomorphism of the closed disk that is K-quasiconformal in the interior and fixes the boundary $\partial\mathbb{D}$ pointwise. Show that there exists a function $\eta(K)$ (independent of f) such that $|f(0)| \leq \eta(K)$, and such that $\eta(K) \to 0$ as $K \to 1$.