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Computation of the Lyapunov spectrum for continuous-time dynamical systems and discrete maps

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In this paper, we describe in detail a method of computing Lyapunov exponents for a continuous-time dynamical system and extend the method to discrete maps. Using this method, a partial Lyapunov spectrum can be computed using fewer equations as compared to the computation of the full spectrum, there is no difficulty in evaluating degenerate Lyapunov spectra, the equations are straightforward to generalize to higher dimensions, and the minimal set of dynamical variables is used. Explicit proofs and other details not given in previous work are included here. [S1063-651X(99)07212-8]

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I. INTRODUCTION

Chaos plays an important role in a large class of dynamical systems. The question of detecting and quantifying chaos has therefore become an important one. The spectrum of Lyapunov exponents has proven to be the most useful dynamical diagnostic for chaotic systems [1] and several methods exist for computing these exponents [1–5]. However, no single method appears to be optimal. For example, QR and singular value decomposition (SVD) methods [2,3] require frequent renormalization (to combat exponential growth of the separation vector between the fiducial and nearby trajectories) and reorthogonalization (to overcome the exponential collapse of initially orthogonal separation vectors onto the direction of maximal growth). The existing continuous versions of the QR and SVD methods also suffer from the additional disadvantage of being unable to compute the partial Lyapunov spectrum using a fewer number of equations/operations than required for the computation of the full spectrum [3]. Further, the continuous SVD method breaks down when computing degenerate Lyapunov spectra [3]. The symplectic method [4] is applicable only to Hamiltonian systems (and a few generalizations thereof) and has proven difficult to extend to systems of moderate size, though this is possible in principle [6]. It also does not permit easy evaluation of partial Lyapunov spectra.

In an earlier paper [7], we had briefly outlined a method for computing Lyapunov exponents for continuous-time dynamical systems. We proposed a general method which analytically obviates the need for rescaling and reorthogonalization. Our method did away with the other shortcomings listed above: A partial Lyapunov spectrum could be computed using fewer equations as compared to the computation of the full spectrum, there was no difficulty in evaluating degenerate Lyapunov spectra, the equations were straightforward to generalize to higher dimensions, and the method used the minimal set of dynamical variables. Since our method was based on exact differential equations for the Lyapunov exponents, global invariances of the Lyapunov spectrum were preserved in principle.

In the present paper, we describe in detail the above method for continuous-time dynamical systems. In the earlier paper, some of the advantages of our method were merely stated without any proofs. Here we provide analytical proofs of these statements; these are given in the Appendixes since they are quite involved. In the earlier paper we had considered only two- and three-dimensional examples. In this paper, we extend this up to six dimensions. More importantly, we generalize our method to discrete maps while retaining all the advantages listed above.

II. CONTINUOUS-TIME DYNAMICAL SYSTEMS

We briefly recall the method presented in our earlier paper [7]. Consider an n-dimensional continuous-time dynamical system,

$$\frac{d\mathbf{z}}{dt}=\mathbf{F}(\mathbf{z},t), \tag{2.1}$$

where $\mathbf{z}=(z_1,z_2,\ldots,z_n)$ and $\mathbf{F}$ is an n-dimensional vector field. Let $\mathbf{Z}(t)=\mathbf{z}(t)-\mathbf{z}_0(t)$ denote deviations from the fiducial trajectory $\mathbf{z}_0(t)$. Linearizing Eq. (2.1) around this trajectory, we obtain

$$\frac{d\mathbf{Z}}{dt}=\mathbf{DF}(\mathbf{z}_0(t),t)\cdot\mathbf{Z}, \tag{2.2}$$

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where **DF** denotes the $n\times n$ Jacobian matrix.

Integrating the linearized equations along the fiducial trajectory yields the tangent map $M(\mathbf{z}_0(t),t)$ which takes the initial variables $\mathbf{Z}^{\rm in}$ into the time-evolved variables $\mathbf{Z}(t)=M(t)\mathbf{Z}^{\rm in}$ [the dependence of M on the fiducial trajectory $\mathbf{z}_0(t)$ is understood]. Let Λ be an $n\times n$ matrix given by $\Lambda=\lim_{t\to\infty}(M\tilde{M})^{1/2t}$, where $\tilde{M}$ denotes the matrix transpose of M. The Lyapunov exponents then equal the logarithm of the eigenvalues of Λ [1].

It is clear that M is of central importance in the evaluation of Lyapunov exponents. Its evolution equation can be easily derived:

$$\frac{dM}{dt}=\mathbf{DF}M. \tag{2.3}$$

As is well known [8], the matrix M can be written as the product $M=QR$ of an orthogonal $n\times n$ matrix Q and an upper-triangular $n\times n$ matrix R with positive diagonal entries. Substituting this into Eq. (2.3), we obtain

$$\dot{Q}R+Q\dot{R}=\mathbf{DF}QR, \tag{2.4}$$

where the overdot denotes a time derivative. Multiplying the above equation by $\tilde{Q}$ from the left and R^{-1} from the right, we get

$$\tilde{Q}\dot{Q}+\dot{R}R^{-1}=\tilde{Q}\mathbf{DF}Q. \tag{2.5}$$

Note that $\tilde{Q}\dot{Q}$ is a skew (anti)symmetric matrix for any orthogonal matrix Q and $\dot{R}R^{-1}$ is still an upper-triangular matrix.

In our method, we employ an easy to obtain explicit representation of the orthogonal matrix Q from group representation theory [9]. One advantage is that a minimum number of variables is used to characterize the system: $n(n-1)/2$ in Q and further n variables in R, for a total of $n(n+1)/2$. The matrix Q is represented as a product of $n(n-1)/2$ orthogonal matrices, each of which corresponds to a simple rotation in the (i-jth plane ($i<j$) [9]. Denoting the matrix corresponding to this rotation by $Q^{(ij)}$, its matrix elements are given by

$$\begin{aligned} Q_{kl}^{(ij)}&=1 \quad \text{if} \quad k=l\neq i,j\\ &=\cos\phi \quad \text{if} \quad k=l=i \quad \text{or} \quad j\\ &=\sin\phi \quad \text{if} \quad k=i,\ \ l=j\\ &=-\sin\phi \quad \text{if} \quad k=j, \quad l=i\\ &=0 \quad \text{otherwise}. \end{aligned} \tag{2.6}$$

Here ϕ denotes an angle variable. Thus, the $n\times n$ matrix Q is represented by

$$Q=Q^{(12)}Q^{(13)}\cdots Q^{(1n)}Q^{(23)}\cdots Q^{(n-1,n)}. \tag{2.7}$$

Hence Q is parametrized by $n(n-1)/2$ angles which we denote by θ_i $(i=1,\ldots,n(n-1)/2)$. These angles will be collectively denoted by θ.

Here Q is represented by a special orthogonal matrix (with determinant equal to $+1$) because of the choice of initial conditions. We choose the identity matrix as the initial orthogonal matrix. That is, we start with a matrix from the SO(n) component of the group of orthogonal matrices. Since we are dealing with continuous-time dynamical systems for the present, due to continuity, we remain in the same component for all time. Hence, we are justified in choosing Q to be a SO(n) matrix. For large values of n, directly using Eq. (2.7) to obtain the representation of Q can be cumbersome: In Appendix A, we give a prescription for calculating the elements of a SO(n) matrix in a more direct fashion.

Since the upper-triangular matrix R has positive diagonal entries, it can be represented as follows:

$$R=\begin{pmatrix} e^{\lambda_1} & r_{12} & \cdots & \cdots & r_{1n}\\ 0 & e^{\lambda_2} & r_{23} & \cdots & r_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & e^{\lambda_n} \end{pmatrix}. \tag{2.8}$$

The quantities λ_i will be shown to be intimately related to the Lyapunov exponents. Our final equations will be in terms of the λ_i which already appear in the exponent, thus removing the need for rescaling. The quantities r_{ij} represent the supradiagonal terms in R.

Using the above representation of R, we obtain

$$\dot{R}R^{-1}=\begin{pmatrix} \dot{\lambda}_1 & r'_{12} & \cdots & \cdots & r'_{1n}\\ 0 & \dot{\lambda}_2 & r'_{23} & \cdots & r'_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dot{\lambda}_n \end{pmatrix}. \tag{2.9}$$

The quantities r'_{ij} are of no concern since they are not present in the final equations.

Substituting the above expression in Eq. (2.5) and comparing diagonal terms on both sides, we obtain

$$\dot{\lambda}_i=S_{ii}, \quad i=1,2,\ldots,n, \tag{2.10}$$

where $S\equiv\tilde{Q}\mathbf{DF}Q$. This is easily seen to be true since $\tilde{Q}\dot{Q}$ is an antisymmetric matrix with diagonal entries zero and $\dot{R}R^{-1}$ has $\dot{\lambda}_i$ as the diagonal entries. The Lyapunov exponents are given by [3]

$$\lim_{t\to\infty}\frac{\lambda_i}{t}. \tag{2.11}$$

In general, in the limit $t\to\infty$ the Lyapunov exponents constitute a monotically decreasing sequence. Thus, the Lyapunov exponents can be obtained by solving the differential equations given in Eq. (2.10) for long times. However, since the right-hand side depends on the angles θ_i, we also require differential equations governing the evolution of these angles.

Differential equations for the angles can be obtained by comparing the subdiagonal elements in Eq. (2.5). Since $\dot{R}R^{-1}$ has zero subdiagonal entries, this gives

$$(\tilde{Q}\dot{Q})_{21}=S_{21},(\tilde{Q}\dot{Q})_{31}=S_{31},\ldots,(\tilde{Q}\dot{Q})_{n,n-1}=S_{n,n-1}. \tag{2.12}$$

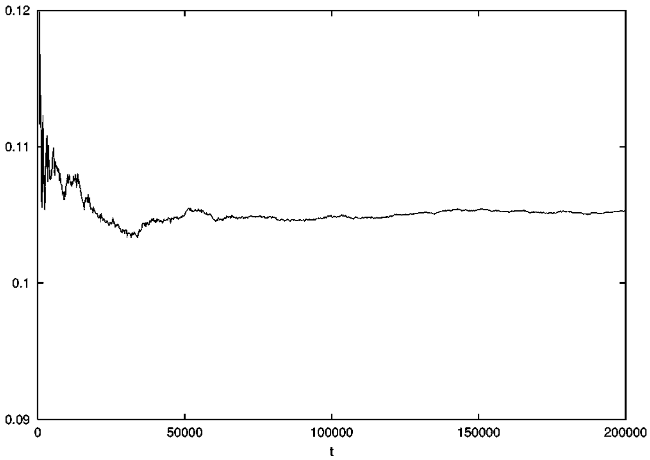

FIG. 1. This figure shows the evolution of λ_1/t as a function of time t for the driven van der Pol oscillator. The parameter values used are $d=-5$, $b=5$, and $w=2.466$.

Note that the above equations are decoupled from the equations for λ_i. This avoids potential problems with degenerate Lyapunov spectra. The above set of differential equations for the angles can be easily set in the following more standard form:

$$\dot{\theta}_i = g_i(\theta), \quad i=1,2,\ldots,n(n-1)/2. \tag{2.13}$$

Equations (2.10) and (2.13) form a system of $n(n+1)/2$ ordinary differential equations that can be solved to obtain the Lyapunov exponents. Since the differential equations are for the angles and not directly for the matrix elements of Q, numerical errors can never lead to loss of orthogonality. Consequently, the need for reorthogonalization is obviated in our method.

Our system of differential equations has another attractive feature. The equation for λ_1 depends only on the first $(n-1)$ θ_i's. Therefore, if one is interested in only the largest Lyapunov exponent, one needs to solve only n equations [as opposed to $n(n+1)/2$ for the full spectrum]. The equation for λ_2 depends only on the first $(2n-3)$ θ_i's. Therefore, to obtain the first two Lyapunov exponents, one needs to solve only $(2n-1)$ equations. In general, to solve for the first m Lyapunov exponents, one has to solve $m(2n-m+1)/2$ equations which is always less than $n(n+1)/2$ (the total number of equations) for $m<n$. This is in contrast to the situation for the conventional continuous QR or SVD methods, where it is computationally costlier to evaluate a partial spectrum once a threshold is crossed [3]. The proof of the above important statement is quite involved: It can be found in Appendix B.

Another interesting feature of this method is the following:

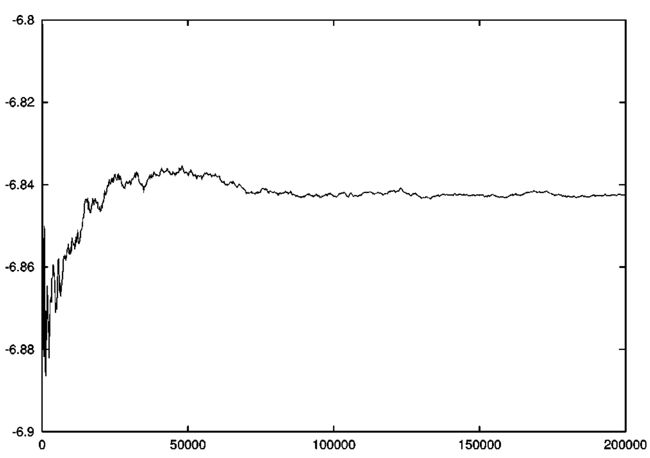

FIG. 2. This figure shows the evolution of λ_2/t as a function of time t for the driven van der Pol oscillator. The parameter values used are $d=-5$, $b=5$, and $w=2.466$.

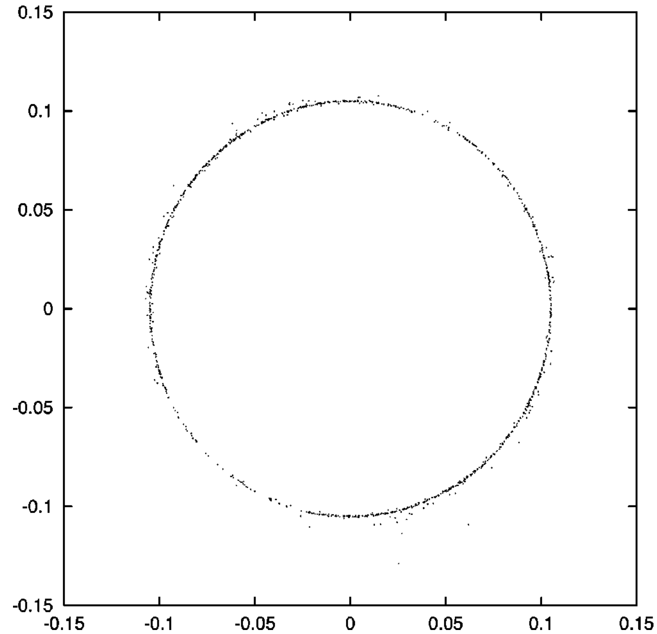

FIG. 3. This figure exhibits the polar plot of λ_1/t as a function of $\theta(t)$ for the driven van der Pol oscillator. The parameter values used are $d=-5$, $b=5$, and $w=2.466$.

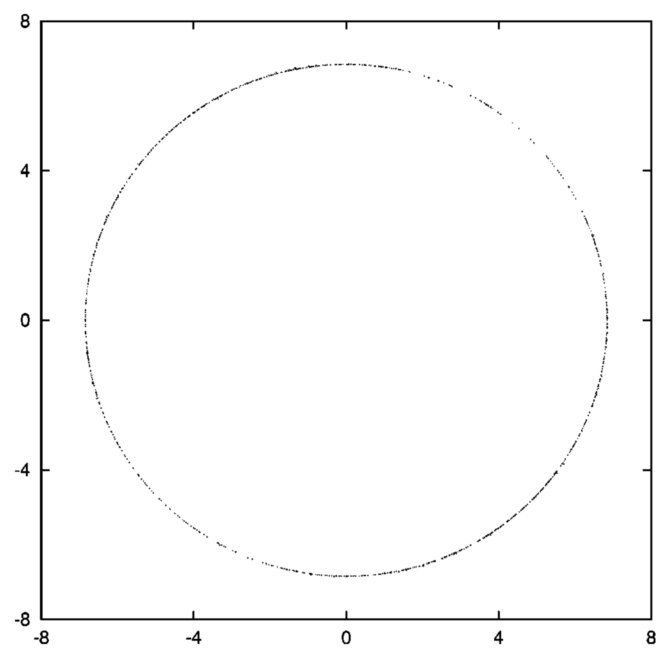

FIG. 4. This figure exhibits the polar plot of λ_2/t as a function of $\theta(t)$ for the driven van der Pol oscillator. The parameter values used are $d=-5$, $b=5$, and $w=2.466$.

$$\sum_{i=1}^{n} \dot{\lambda}_i = \sum_{i=1}^{n} (\tilde{Q}\mathbf{DF}Q)_{ii} \tag{2.14}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} Q_{ji}\mathbf{DF}_{jk}Q_{ki} \tag{2.15}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbf{DF}_{jk}\left(\sum_{i=1}^{n} Q_{ji}Q_{ki}\right). \tag{2.16}$$

Since $Q\tilde{Q}=I$ (Q being an orthogonal matrix), $\Sigma_{i=1}^{n} Q_{ji}Q_{ki} = \delta_{jk}$, $i,j=1,2,\ldots,n$. Therefore, we get

$$\dot{\lambda}_1 + \dot{\lambda}_2 + \cdots + \dot{\lambda}_n = \mathbf{DF}_{11} + \mathbf{DF}_{22} + \cdots + \mathbf{DF}_{nn}. \tag{2.17}$$

This relation can be used to speed up the numerical integration of the differential equation for λ_n.

To illustrate the application of the method to a system with two degrees of freedom, we consider the driven van der Pol oscillator:

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= -d(1-z_1^2)z_2 - z_1 + b\cos wt. \end{aligned} \tag{2.18}$$

We have already considered this system in our earlier paper [7]. Here we present more detailed results for the system. For the parameter values $d=-5$, $b=5$, and $w=2.466$, the results for the Lyapunov exponents are shown in Figs. 1 and 2. The results are in agreement with values obtained by the existing methods. In Fig. 3, we plot λ_1/t as a function of $\theta(t)$ in a polar plot. The figure shows that we obtain a circle asymptotically. This suggests that our variables are akin to the ''action-angle'' variables encountered in classical mechanics. At present, methods for exploiting this feature to speed up the convergence rate of the Lyapunov exponents are being investigated. In Fig. 4, we exhibit the polar plot of λ_2/t versus θ. Here, the circle is approached even faster asymptotically.

For the $n=4$ case, we have to generalize the equations given in our earlier paper [7]. The dominant Lyapunov exponent for the $n=4$ case is given by integrating the following equation numerically (along with equations for θ_1, θ_2, and θ_3 which we have not included below) for long times and dividing by time:

$$\begin{aligned} \dot{\lambda}_1 = {} & \cos^2\theta_1 \cos^2\theta_2 \cos^2\theta_3 df_{11} + \sin^2\theta_1 \cos^2\theta_2 \cos^2\theta_3 df_{22} + \sin^2\theta_2 \cos^2\theta_3 df_{33} + \sin^2\theta_3 df_{44} \\ & - \frac{1}{2}\sin 2\theta_1 \cos^2\theta_2 \cos^2\theta_3 (df_{12}+df_{21}) - \frac{1}{2}\cos\theta_1 \sin 2\theta_2 \cos^2\theta_3 (df_{13}+df_{31}) - \frac{1}{2}\cos\theta_1 \cos\theta_2 \sin 2\theta_3 (df_{14}+df_{41}) \\ & + \frac{1}{2}\sin\theta_1 \sin 2\theta_2 \cos^2\theta_3 (df_{23}+df_{32}) + \frac{1}{2}\sin\theta_1 \cos\theta_2 \sin 2\theta_3 (df_{24}+df_{42}) + \frac{1}{2}\sin\theta_2 \sin 2\theta_3 (df_{34}+df_{43}). \end{aligned} \tag{2.19}$$

We apply this method to two systems. We start with an example of a system with the Hamiltonian

$$H = \frac{p_x^2+p_y^2}{2} + \frac{x^2y^2}{2} + \frac{x^4+y^4}{32}. \tag{2.20}$$

The Hamilton equations of motion are

$$\dot{x} = \frac{\partial}{\partial p_x} H = p_x, \tag{2.21}$$

$$\dot{p}_x = -\frac{\partial}{\partial x} H = -(xy^2 + x^3/8), \tag{2.22}$$

$$\dot{y} = \frac{\partial}{\partial p_y} H = p_y, \tag{2.23}$$

$$\dot{p}_y = -\frac{\partial}{\partial y} H = -(x^2y + y^3/8). \tag{2.24}$$

Two of the Lyapunov exponents tend to zero and the other two are the negative of each other, as expected. The second example is given by

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= -d(1-z_1^2)z_2 - z_1 + b\cos wt, \\ z_3' &= z_4, \\ z_4' &= -d(1-z_3^2)z_4 - z_3 + b\cos wt, \end{aligned} \tag{2.25}$$

where the parameter values are the same as in the $n=2$ case. The values of the Lyapunov exponents of the above set of equations, obtained by our method, are found to be the same as those of the van der Pol oscillator, repeated twice, as expected.

Our method has been further extended to the case $n=6$. It has been applied to a generalization of the example given in Eq. (2.20) to three degrees of freedom. Results obtained are as expected.

III. LYAPUNOV EXPONENTS FOR DISCRETE MAPS

In the preceding section, we considered our method as applied to continuous-time dynamical systems. In this section, we generalize our method of computing Lyapunov exponents to discrete maps.

Let us consider the following nonlinear map:

$$\mathbf{z}(n+1)=\mathbf{F}(\mathbf{z}(n)), \tag{3.1}$$

where $\mathbf{F}$: $\mathbb{R}m\to\mathbb{R}^m$ is a differentiable vector function and $\mathbf{z}$ is an m vector. Let the matrix $(\partial F_i/\partial z_j)$ of partial derivatives of the components F_i at $\mathbf{z}$ be denoted by $\mathbf{DF}(\mathbf{z})$. Then the corresponding matrix of partial derivatives for the nth iterate $\mathbf{F}^n$ of $\mathbf{F}$ is given by

$$\frac{\partial(\mathbf{F}^n)_i}{\partial z_j}=\mathbf{DF}(\mathbf{F}^{n-1}\mathbf{z})\cdots\mathbf{DF}(\mathbf{F}(\mathbf{z}))\mathbf{DF}(\mathbf{z}). \tag{3.2}$$

Let

$$A_n=\mathbf{DF}(\mathbf{F}^{n-1}\mathbf{z})\cdots\mathbf{DF}(\mathbf{F}(\mathbf{z}))\mathbf{DF}(\mathbf{z}), \tag{3.3}$$

which implies

$$A_n=\mathbf{DF}(\mathbf{z}(n-1))\cdots\mathbf{DF}(\mathbf{z}(1))\mathbf{DF}(\mathbf{z}(0)). \tag{3.4}$$

Then

$$A_{n+1}=\mathbf{DF}(\mathbf{z}(n))A_n. \tag{3.5}$$

Similar to the continuous case, A_n can be decomposed into a product of an orthogonal matrix and an upper-triangular matrix with positive diagonal elements. That is,

$$Q_{n+1}R_{n+1}=\mathbf{DF}(\mathbf{z}(n))Q_nR_n, \tag{3.6}$$

where $Q_n,Q_{n+1}\in \mathrm{O}(m)$ and R_n and R_{n+1} are the upper-triangular matrices with positive diagonal elements. Let $\{R_{ii}^{(n)},i=1,\ldots,m\}$ be the set of diagonal elements of R_n. Then

$$\lim_{n\to\infty}\frac{\ln R_{ii}^{(n)}}{n}=\lambda_i',$$

where $\{\lambda_i',i=1,\ldots,m\}$ are the Lyapunov exponents.

From Eq. (3.6), we have

$$R_{n+1}R_n^{-1}=\tilde{Q}_{n+1}\mathbf{DF}Q_n, \tag{3.7}$$

where the left-hand side is an upper-triangular matrix with $R_{ii}^{(n+1)}/R_{ii}^{(n)}$ $(i=1,2,\ldots,m)$ as the diagonal elements. The dependence of $\mathbf{DF}$ on $\mathbf{z}(n)$ is understood. Since $R_{ii}^{(j)}$ $(j=n,n+1;i=1,2,3,\ldots,m)$ are positive, they can be represented by $\exp\lambda_i^{(j)}(j=n,n+1;\ i=1,2,3,\ldots,m)$. Therefore, the diagonal elements of the matrix $R_{n+1}R_n^{-1}$ are given by $\exp\lambda_i^{(n+1)}-\lambda_i^{(n)}$, $i=1,2,3,\ldots,m$.

From Eq. (3.7), the equations for $\theta_i^{(n+1)}$'s are given by the following set of equations (since $R_{n+1}R_n^{-1}$ is an upper triangular matrix):

$$(\tilde{Q}_{n+1}\mathbf{DF}Q_n)_{lk}=0,\quad l>k, \tag{3.8}$$

where $l=2,3,\ldots,m$ and $k=1,2,3,\ldots,l-1$. To solve these equations for $\theta_i^{(n+1)}$'s, we have to first parametrize the Q_n's. For continuous-time dynamical systems, because of continuity arguments we were able to parametrize Q_n as a SO(n) matrix. On the other hand, in the discrete case, the Q_n's may belong to either of the following: SO(m) or the component with determinant -1 [denoted by O$'(m)$]. Therefore, the following four combinations have to be taken into account while developing the algorithm for computing the angles and subsequently the Lyapunov exponents: (i) $Q_n\in$ SO(m) and $Q_{n+1}\in$ SO(m), (ii) $Q_n\in$ SO(m) and $Q_{n+1}\in$ O$'(m)$, (iii) $Q_n\in$ O$'(m)$ and $Q_{n+1}\in$ SO(m), (iv) $Q_n\in$ O$'(m)$ and $Q_{n+1}\in$ O$'(m)$.

To take into account the above four cases, we define

$$Q_j=Q_j'P_j,\quad j=n,n+1, \tag{3.9}$$

where $Q_j'\in$ SO(m) is parametrized using $\theta_i^{(j)}$ $(j=n,n+1;\ i=1,2,\ldots,m(m-1)/2)$ [cf. Eq. (2.7)]. The $m\times m$ matrices $P_j(j=n,n+1)$ are diagonal matrices with diagonal elements $(P_j)_{kk}=1$ (or -1), $k=1,2,\ldots,m$. Thus, if $Q_j\in$ SO(m), then P_j has zero or an even number of -1's and if $Q_j\in$ O$'(m)$, then P_j has an odd number of -1's.

For computation of the angles, we now show that we can still parametrize Q_n's as SO(n) matrices even in the discrete case. From Eq. (3.8), we have [using Eq. (3.9) for Q_n]

$$\begin{aligned}
(\tilde{Q}_{n+1}\mathbf{DF}Q_n)_{lk}
&=\sum_{r=1}^{m}(\tilde{Q}_{n+1})_{lr}(\mathbf{DF}Q_n)_{rk}\\
&=\sum_{r,s=1}^{m}(Q_{n+1}'P_{n+1})_{rl}\mathbf{DF}_{rs}(Q_n'P_n)_{sk}\\
&=\sum_{r,s,u,v=1}^{m}(Q_{n+1}')_{ru}(P_{n+1})_{ul}\mathbf{DF}_{rs}(Q_n')_{sv}(P_n)_{vk}\\
&=\sum_{r,s,u,v=1}^{m}(Q_{n+1}')_{rl}(P_{n+1})_{ll}\mathbf{DF}_{rs}(Q_n')_{sk}(P_n)_{kk}
\end{aligned}$$

$$[\text{since } (P_n)_{ij}=0\ \ \forall\ \ i\neq j]$$

$$\begin{aligned}
&=(P_{n+1})_{ll}(P_n)_{kk}\sum_{r,s=1}^{m}(\tilde{Q}_{n+1}')_{rl}\mathbf{DF}_{rs}(Q_n')_{sk}\\
&=(P_{n+1})_{ll}(P_n)_{kk}(\tilde{Q}_{n+1}'\mathbf{DF}Q_n')_{lk}.
\end{aligned} \tag{3.10}$$

Thus,

$$(\tilde{Q}_{n+1}\mathbf{DF}Q_n)_{lk}=0, \tag{3.11}$$

which implies

$$(\tilde{Q}_{n+1}'\mathbf{DF}Q_n')_{lk}=0, \tag{3.12}$$

where $l>k$, $l=2,3,\ldots,m$, and $k=1,2,\ldots,l-1$. Therefore, for solving for $\theta_j^{(n+1)}$'s $[j=1,2,\ldots,m(m-1)/2]$, it is sufficient to solve Eq. (3.12) instead of Eq. (3.8), irrespective of whether the Q_j's belong to SO(m) or O'(m).

But the Q_j's do matter while deriving the equations for $\lambda_j^{(n+1)}$'s. To compute these, we compare the diagonal elements in Eq. (3.7) and substitute Eq. (3.9) for Q_{n+1},

$$\begin{aligned}
\exp \lambda_j^{(n+1)} - \lambda_j^{(n)} &= (\tilde{Q}_{n+1}\mathbf{DF}Q_n)_{jj} \\
&= \sum_{r=1}^{m} (Q_{n+1})_{rj}(\mathbf{DF}Q_n)_{rj} \\
&= \sum_{r=1}^{m} (Q'_{n+1}P_{n+1})_{rj}(\mathbf{DF}Q_n)_{rj} \\
&= \sum_{r,u=1}^{m} (Q'_{n+1})_{ru}(P_{n+1})_{uj}(\mathbf{DF}Q_n)_{rj} \\
&= \sum_{r=1}^{m} (Q'_{n+1})_{rj}(P_{n+1})_{jj}(\mathbf{DF}Q_n)_{rj} \\
&\qquad [\text{since } (P_{n+1})_{uj} = 0 \forall \quad u \neq j] \\
&= (P_{n+1})_{jj}\left(\sum_{r=1}^{m} (Q'_{n+1})_{rj}(\mathbf{DF}Q_n)_{rj} \right) \\
&= (P_{n+1})_{jj}(\tilde{Q}'_{n+1}\mathbf{DF}Q_n)_{jj}, \qquad (3.13)
\end{aligned}$$

where Q_n and $\mathbf{DF}$ are already known and Q'_{n+1} is also known from Eq. (3.12) ($\theta_j^{(n+1)}$'s are the angle variables of Q'_{n+1}).

If $(\tilde{Q}'_{n+1}\mathbf{DF}Q_n)_{jj} < 0$ (>0), then $(P_{n+1})_{jj} = -1(1)$. But this amounts to taking the absolute value of the right-hand side of Eq. (3.13). Thus,

$$\exp \lambda_j^{(n+1)} - \lambda_j^{(n)} = |(\tilde{Q}'_{n+1}\mathbf{DF}Q_n)_{jj}|. \qquad (3.14)$$

But [using Eq. (3.9) for Q_n]

$$\begin{aligned}
|(\tilde{Q}'_{n+1}DFQ_n)_{jj}| &= \left| \sum_{r=1}^{m} (\tilde{Q}'_{n+1}\mathbf{DF})_{jr}(Q'_nP_n)_{rj} \right| \\
&= \left| \sum_{r,s=1}^{m} (\tilde{Q}'_{n+1}\mathbf{DF})_{jr}(Q'_n)_{rs}(P_n)_{sj} \right| \\
&= \left| (P_n)_{jj}\left(\sum_{r=1}^{m} (\tilde{Q}'_{n+1}\mathbf{DF})_{jr}(Q'_n)_{rj} \right| \right) \\
&\qquad [\text{since } (P_n)_{sj} = 0 \forall \quad s \neq j] \\
&= |(P_n)_{jj}(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{jj}| \\
&= |(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{jj}|, \qquad (3.15)
\end{aligned}$$

where we have used the fact that $(P_n)_{jj}$ is equal to 1 or -1 in the last step.

Thus, for finding $\lambda_j^{(n+1)}$'s $(j=1,\ldots,m)$, we have to solve the following m equations:

$$\lambda_j^{(n+1)} = \lambda_j^{(n)} + \ln|(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{jj}| \quad \text{for} \quad j=1,2,\ldots,m \qquad (3.16)$$

with the following $m(m-1)/2$ equations for $\theta^{(n+1)}$'s:

$$(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{lk} = 0, \quad l>k, \qquad (3.17)$$

where $l=2,3,\ldots,m$, $k=1,2,\ldots,l-1$, and where Q'_{n+1}, Q'_n, and SO(m) are matrices.

We illustrate the working of this method by taking $m=2$. The SO(2) matrices Q'_i $(i=n,n+1)$ are parametrized as

$$\begin{pmatrix} \cos\theta^{(i)} & \sin\theta^{(i)} \\ -\sin\theta^{(i)} & \cos\theta^{(i)} \end{pmatrix}. \qquad (3.18)$$

Further, let $\mathbf{DF}$ be parametrized as

$$\mathbf{DF} = \begin{pmatrix} df_{11} & df_{12} \\ df_{21} & df_{22} \end{pmatrix}, \qquad (3.19)$$

and let R_n be parametrized as (using λ and μ for notational simplicity)

$$R_n = \begin{pmatrix} e^{\lambda^{(n)}} & * \\ 0 & e^{\mu^{(n)}} \end{pmatrix}. \qquad (3.20)$$

Here * is used to denote quantities which we are not interested in and which will not enter into our final expressions.

Substituting the above representations in Eq. (3.6), we get

$$\begin{aligned}
&\begin{pmatrix} \cos\theta^{(n+1)} & * \\ -\sin\theta^{(n+1)} & * \end{pmatrix}\begin{pmatrix} e^{\lambda^{(n+1)}} & * \\ 0 & e^{\mu^{(n+1)}} \end{pmatrix} \\
&= \begin{pmatrix} df_{11} & df_{12} \\ df_{21} & df_{22} \end{pmatrix}\begin{pmatrix} \cos\theta^{(n)} & * \\ -\sin\theta^{(n)} & * \end{pmatrix}\begin{pmatrix} e^{\lambda^{(n)}} & * \\ 0 & e^{\mu^{(n)}} \end{pmatrix}.
\end{aligned}$$

Carrying out the matrix multiplications, we get

$$\begin{aligned}
&\begin{pmatrix} \cos\theta^{(n+1)}e^{\lambda^{(n+1)}} & * \\ -\sin\theta^{(n+1)}e^{\lambda^{(n+1)}} & * \end{pmatrix} \\
&= \begin{pmatrix} (df_{11}\cos\theta^{(n)} - df_{12}\sin\theta^{(n)})e^{\lambda^{(n)}} & * \\ -(df_{22}\sin\theta^{(n)} - df_{21}\cos\theta^{(n)})e^{\lambda^{(n)}} & * \end{pmatrix}. \qquad (3.21)
\end{aligned}$$

That is,

$$\sin\theta^{(n+1)}e^{\lambda^{(n+1)}} = (df_{22}\sin\theta^{(n)} - df_{21}\cos\theta^{(n)})e^{\lambda^{(n)}}, \qquad (3.22)$$

$$\cos\theta^{(n+1)}e^{\lambda^{(n+1)}} = (df_{11}\cos\theta^{(n)} - df_{12}\sin\theta^{(n)})e^{\lambda^{(n)}}. \qquad (3.23)$$

Dividing Eq. (3.22) by Eq. (3.23), we have

$$\theta^{(n+1)} = \tan^{-1}\left(\frac{df_{22}\sin\theta^{(n)} - df_{21}\cos\theta^{(n)}}{df_{11}\cos\theta^{(n)} - df_{12}\sin\theta^{(n)}} \right). \qquad (3.24)$$

The above equations are used to calculate the $\theta^{(n+1)}$'s. Consequently, the matrix $Q'_{(n+1)}$ is also fully determined.

We are now in a position to calculate $\lambda^{(n+1)}$ and $\mu^{(n+1)}$. We have from Eq. (3.7),

$$\begin{pmatrix} e^{\lambda^{(n+1)}-\lambda^{(n)}} & * \\ 0 & e^{\mu^{(n+1)}-\mu^{(n)}} \end{pmatrix} = \begin{pmatrix} (\tilde{Q}_{n+1}\mathbf{DF}Q_n)_{11} & * \\ 0 & (\tilde{Q}_{n+1}\mathbf{DF}Q_n)_{22} \end{pmatrix}. \tag{3.25}$$

Equating the elements of the matrices on both sides and using the arguments given prior to Eq. (3.16), we have

$$\lambda^{(n+1)} = \lambda^{(n)} + \ln|(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{11}|, \tag{3.26}$$

$$\mu^{(n+1)} = \mu^{(n)} + \ln|(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{22}|. \tag{3.27}$$

Since Q' depends only on θ, and θ in turn does not depend on either λ or μ, the presence of Q'_{n+1} on the right-hand side of the above equations does not create any problems. The Lyapunov exponents λ' and μ' are given by

$$\lambda' = \lim_{n\to\infty}\frac{\lambda^{(n)}}{n}, \tag{3.28}$$

$$\mu' = \lim_{n\to\infty}\frac{\mu^{(n)}}{n}. \tag{3.29}$$

As an example of the $m=2$ case, we consider the Henon map,

$$x_{n+1} = 1 - a x_n^2 + y_n, \tag{3.30}$$

$$y_{n+1} = b x_n. \tag{3.31}$$

Then $\mathbf{DF}(x_n, y_n)$ is given by

$$\begin{pmatrix} -2ax_n & 1 \\ 0.3 & 0 \end{pmatrix}. \tag{3.32}$$

We have computed the Lyapunov exponents using our method for the parameter values $a=1.4$ and $b=0.3$. After 10 000 iterations, the Lyapunov exponents were found to be

$$\lambda' = 0.4181, \quad \mu' = -1.6221.$$

The results are in agreement with the values obtained by other methods.

The partial decoupling of the equations for the angles and the exponents, seen in the continuous-time case, is observed in the discrete case, too. Equations for the full set of $m(m-1)/2$ $\theta_j^{(n+1)}$'s $[j=1,2,\ldots,m(m-1)/2]$ are given by Eq. (3.12). The first $m-1$ $\theta^{(n+1)}$'s are found from the following $m-1$ equations:

$$(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{i1} = 0, i=2,3,\ldots,m. \tag{3.33}$$

That is,

$$\sum_{k,j=1}^{n} (Q'_{n+1})_{ki}\mathbf{DF}_{kj}(Q'_n)_{j1} = 0, \quad i=2,3,\ldots,m. \tag{3.34}$$

Only the first $(m-1)$ $\theta^{(n)}$'s feature in the above set of equations. On solving these $m-1$ equations, we get the first $m-1$ $\theta^{(n+1)}$'s in terms of the first $m-1$ $\theta^{(n)}$'s, viz., $\theta_1^{(n)}, \theta_2^{(n)}, \ldots, \theta_{m-1}^{(n)}$.

Similarly, the next $(m-2)$ $\theta^{(n+1)}$'s are found by solving the following set of equations:

$$(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{i2} = 0, \quad i=3,4,\ldots,m. \tag{3.35}$$

Only the first $(2m-3)$ $\theta^{(n)}$'s feature in the above set of equations. Here again, all the $\theta^{(n+1)}$'s, except for the first $(2m-3)\,\theta^{(n+1)}$'s, get eliminated so that the resulting equations depend only on the first $(m-1)$ and the next $(m-2)$, i.e., the first $(2m-3)$ $\theta^{(n)}$'s and $\theta^{(n+1)}$'s. On solving these equations for $\theta_m^{(n+1)}, \theta_{m+1}^{(n+1)}, \ldots, \theta_{2m-3}^{(n+1)}$, we observe that the above set of $(m-2)$ $\theta^{(n+1)}$'s depends only on the first $(2m-3)$ $\theta^{(n)}$'s. Similarly, $\theta_{2m-2}^{(n+1)}, \theta_{2m-1}^{(n+1)}, \ldots, \theta_{3m-6}^{(n+1)}$ depend only on the first $(3m-6)$ $\theta^{(n)}$'s and so on. $\theta_{m(m-1)/2-2}^{(n+1)}$ and $\theta_{m(m-1)/2-1}^{(n+1)}$ depends only on the first $[m(m-1)/2]-1$ $\theta^{(n)}$'s. Finally, $\theta_{m(m-1)/2}^{(n+1)}$ is seen to depend on all $m(m-1)/2$ $\theta^{(n)}$'s.

The equations for $\lambda_i^{(n+1)}$'s are given by

$$\lambda_i^{(n+1)} = \lambda_i^n + \ln|(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{ii}|, \quad i=1,\ldots,m, \tag{3.36}$$

and the Lyapunov exponents are given by

$$\lambda'_i = \lim_{n\to\infty}\frac{\lambda_i^{(n)}}{n}, \quad i=1,2,\ldots,m. \tag{3.37}$$

Since

$$(\tilde{Q}'_{n+1}\mathbf{DF}Q'_n)_{ii} = \sum_{k,j=1}^{n}(Q'_{n+1})_{ki}\mathbf{DF}_{kj}(Q'_n)_{ji}, \tag{3.38}$$

$\lambda_1^{(n+1)}$ depends on the first columns of Q_n and Q_{n+1} which, in turn, depend only on the first $(m-1)$ $\theta^{(n)}$'s and $\theta^{(n+1)}$'s respectively. Since the first $(m-1)$ $\theta_{(n+1)}$'s depend only on the first $(m-1)$ $\theta^{(n)}$'s, $\lambda_1^{(n+1)}$ depends only on the first $(m-1)$ $\theta^{(n)}$'s. Similarly, $\lambda_2^{(n+1)}$ depends only on the first $(2m-3)$ $\theta^{(n)}$'s, $\lambda_3^{(n+1)}$ on the first $3m-6$ $\theta^{(n)}$'s and so on. Finally, $\lambda_{m-1}^{(n+1)}$ and $\lambda_m^{(n+1)}$ depend on all the $m(m-1)/2$ $\theta^{(n)}$'s. Therefore, similar to the continuous case, to solve for the first m Lyapunov exponents, one has to solve for only $m(2n-m-1)/2$ equations. The complete proof of the above statements is similar to the continuous-time case (see Appendix B).

To illustrate the application of this method to the $m=3$ case, we consider the following map:

$$x_{n+1} = 1 - a x_n^2 + y_n,$$

$$y_{n+1} = b x_n,$$

$$z_{n+1} = z_n,$$

where the parameter values are same as in the $m=2$ case. As expected, the values of the three Lyapunov exponents are

$$\lambda'_1 = 0.4181, \quad \lambda'_2 = -1.6221, \quad \lambda'_3 = 0.$$

For the $m=4$ case, we consider the following map:

$$x_{n+1}=1-ax_n^2+y_n,$$

$$y_{n+1}=bx_n,$$

$$z_{n+1}=1-az_n^2+w_n,$$

$$w_{n+1}=bz_n.$$

For the same parameter values as in the above cases, the Lyapunov exponents are found to be the same as those of the Henon map, repeated twice.

IV. CONCLUSIONS

In this paper, we have described in detail a technique for computing Lyapunov exponents of continuous-time dynamical systems as well as for discrete maps. This method has several advantages over the existing methods. The minimal number of variables is used in the equations and the need for rescaling and reorthonormalization is eliminated. There are no difficulties in calculating degenerate spectra, and global invariances of the Lyapunov spectrum can be preserved [7]. Furthermore, the final set of equations is reduced to a convenient form, simplifying generalization to higher orders. Another major advantage of this method is in the evaluation of partial spectra. Fewer equations/operations are required than for the full spectra, unlike some of the other existing methods. Finally, this method is easily adapted to discrete maps, while retaining all the advantages of the continuous-time case.

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APPENDIX A

In this appendix, we shall give direct expressions for the elements of a general SO(n), $n\geqslant 3$ matrix.

To define a general SO(n) matrix, we first need to define some auxiliary $[n-(k-1)]\times[n-(k-1)]$ matrices denoted by R^k $(k=1,2,\ldots,n-1)$. The matrices R^k's, $k=1,2,\ldots,n-2$, are described below. Elements of the first row are given by

$$R_{11}^k=\prod_{r=m(n,k)}^{p(n,k)}\cos\theta_r, \tag{A1}$$

$$R_{12}^k=\sin\theta_{m(n,k)}, \tag{A2}$$

where

$$m(n,k)=(k-1)(2n-k)/2, \quad p(n,k)=k(2n-k-1)/2. \tag{A3}$$

For $j=3,\ldots,n-(k-1)$,

$$R_{1j}^k=\left(\prod_{r=0}^{j-2}\cos\theta_{m(n,k)+r}\right)\sin\theta_{m(n,k)+j-2}. \tag{A4}$$

Elements of the second row are given by

$$R_{2j}^k=\frac{\partial}{\partial\theta_{m(n,k)}}(R_{1j}^k), \quad j=1,2,\ldots,n-(k-1). \tag{A5}$$

Elements of the remaining rows are given as follows. For $i=3,\ldots,n-(k-1)$ and $j=1,2,\ldots,n-(k-1)$,

$$R_{ij}^k=\frac{\partial}{\partial\theta_{m(n,k)+i-2}}(\mathcal{R}_{ij}), \tag{A6}$$

where $\mathcal{R}_{ij}^k$ is the coefficient of $\Pi_{r=0}^{i-3}\cos\theta_{m(n,k)+r}$ in R_{1j}^k.

Finally, the 2×2 R^{n-1} matrix is given by

$$R_{12}^{n-1}=-R_{21}^{n-1}=\sin\theta_{p(n,k)}, \tag{A7}$$

$$R_{11}^{n-1}=R_{22}^{n-1}=\cos\theta_{p(n,k)}. \tag{A8}$$

Now we are in a position to give direct expressions for the elements of the $n\times n$ matrix $Q\in$ SO(n). The element $Q_{1,n}$ is given by

$$Q_{1n}=\left(\prod_{k=1}^{n-2}\sum_{j_k=2}^{n-(k-1)}\right)\left(\prod_{m=0}^{n-2}R_{j_m-1,j_{m+1}}^{m+1}\right), \tag{A9}$$

where $j_{n-1}=2$ and $j_0=2$. Here we have used the notation $(\Pi_{k=1}^{n-2}\Sigma_{j_k=2}^{n-(k-1)})=\Sigma_{j_1=2}^{n}\Sigma_{j_2=2}^{n-1}\cdots\Sigma_{j_{n-2}=2}^{3}$ (the product symbol in the preceding expression is used only for notational convenience). The other elements in the first row are given by

$$Q_{1,n-1}=\frac{\partial}{\partial\theta_{p(n,n-1)}}(Q_{1,n}), \tag{A10}$$

$$Q_{1,l}=\frac{\partial}{\partial\theta_{p(n,l)}}(\mathcal{Q}_{1,l}), \quad l=1,2,\ldots,n-2, \tag{A11}$$

where $\mathcal{Q}_{1,l}$ is the coefficient of $\Pi_{m=l+1}^{n-1}\cos\theta_{p(n,m)}$ in $Q_{1,n}$.

Elements of the second row of Q are obtained from the expressions

$$Q_{2,l}=\frac{\partial}{\partial\theta_1}(Q_{1,n}), \quad l=1,2,\ldots n. \tag{A12}$$

Elements of the remaining rows can be written as

$$Q_{i,l}=\frac{\partial}{\partial\theta_{i-1}}(\mathcal{Q}_{i,l}), \quad i=3,4,\ldots,n, \quad l=1,2,\ldots,n, \tag{A13}$$

where $\mathcal{Q}_{i,l}$ is the coefficient of $\Pi_{r=1}^{i-2}\cos\theta_r$ in $Q_{1,l}$.

We now apply the above formulas to obtain expressions for a general SO(3) matrix. Setting $n=3$ in the above formulas, we first get [cf. Eq. (A3)]

$$p(3,2)=3, \quad p(3,1)=2, \quad m(3,1)=1. \tag{A14}$$

The last element of the first row is given by [cf. Eq. (A9)]

$$Q_{1,3}=\sum_{j_1=2}^{3} R^1_{1j_1}R^2_{j_1-1,2}$$
$$=R^1_{12}R^2_{12}+R^1_{13}R^2_{22}, \tag{A15}$$

where

$$R^1_{12}=\sin\theta_1, \tag{A16}$$

$$R^1_{13}=\cos\theta_1\sin\theta_2, \tag{A17}$$

$$R^2_{12}=\sin\theta_3, \tag{A18}$$

$$R^2_{22}=\cos\theta_3 \tag{A19}$$

Therefore, the elements of the general SO(3) matrix are

$$Q_{1,3}=\sin\theta_1\sin\theta_3+\cos\theta_1\sin\theta_2\cos\theta_3$$

$$Q_{1,2}=\frac{\partial}{\partial\theta_3}Q_{1,3}=\sin\theta_1\cos\theta_3-\cos\theta_1\sin\theta_2\sin\theta_3,$$

$$Q_{1,1}=\frac{\partial}{\partial\theta_2}(\mathcal{Q}_{1,1})=\frac{\partial}{\partial\theta_2}(\cos\theta_1\sin\theta_2)=\cos\theta_1\cos\theta_2,$$

$$Q_{2,1}=\frac{\partial}{\partial\theta_1}Q_{1,1}=-\sin\theta_1\cos\theta_2,$$

$$Q_{2,2}=\frac{\partial}{\partial\theta_1}Q_{1,2}=\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3, \tag{A20}$$

$$Q_{2,3}=\frac{\partial}{\partial\theta_1}Q_{1,3}=\cos\theta_1\sin\theta_3-\sin\theta_1\sin\theta_2\cos\theta_3,$$

$$Q_{3,1}=\frac{\partial}{\partial\theta_2}(\mathcal{Q}_{3,1})=\frac{\partial}{\partial\theta_2}(\cos\theta_2)=-\sin\theta_2,$$

$$Q_{3,2}=\frac{\partial}{\partial\theta_2}(\mathcal{Q}_{3,2})=\frac{\partial}{\partial\theta_2}(-\sin\theta_2\sin\theta_3)=-\cos\theta_2\sin\theta_3,$$

$$Q_{3,3}=\frac{\partial}{\partial\theta_2}(\mathcal{Q}_{3,3})=\frac{\partial}{\partial\theta_2}(\sin\theta_2\cos\theta_3)=\cos\theta_2\cos\theta_3.$$

These expressions agree with the standard expressions for SO(3) matrix elements as expected.

APPENDIX B

Theorem. To solve for the first m Lyapunov exponents, only $m(2n-m+1)/2$ equations need to be solved.

Proof. We will prove the theorem in stages using a series of lemmas. Consider the orthogonal matrix Q represented by [cf. Eq. (2.7)]

$$Q=Q^{(12)}Q^{(13)}\cdots Q^{(1n)}Q^{(23)}\cdots Q^{(n-1,n)}, \tag{B1}$$

and parametrized by $n(n-1)/2$ angle variables denoted by θ_i $[i=1,2,\ldots,n(n-1)/2]$.

Lemma 1. The elements of the lth column of the orthogonal matrix Q represented as in Eq. (B1) depend only on the first $l(2n-l-1)/2$ θ_i's for $l<n$ and all $n(n-1)/2$ θ_i's for $l=n$.

Proof. Let

$$T_i=Q^{(i,i+1)}Q^{(i,i+2)}\cdots Q^{(i,n)}. \tag{B2}$$

Then,

$$T_1=Q^{(1,2)}Q^{(1,3)}\cdots Q^{(1,n)}. \tag{B3}$$

Since the elements of the matrices $Q^{(1,2)},Q^{(1,3)},\ldots,Q^{(1,n)}$ depend only on the first $(n-1)$ θ_i's, the elements of the matrix T_1 also depend only on these θ_i's, viz., $\theta_1,\theta_2,\ldots,\theta_{n-1}$. And

$$T_2=Q^{(2,3)}Q^{(2,4)}\cdots Q^{(2,n)}. \tag{B4}$$

This matrix is of the form

$$\begin{bmatrix} 1 & & \\ 1 & & 0\cdots 0 \\ 0 & & \\ \vdots & H_2(\theta_n,\theta_{n+1},\ldots,\theta_{2n-3}) & \\ 0 & & \end{bmatrix}. \tag{B5}$$

Here H_2 is an $(n-1)\times(n-1)$ matrix, whose elements depend only on the next $(n-2)$ θ_i's, viz., $\theta_n,\theta_{n+1},\ldots,\theta_{2n-3}$ since the constituent matrices $Q^{(2,3)},Q^{(2,4)},\ldots,Q^{(2,n)}$ depend only on these angles. Note that the first column is just a unit vector.

Continuing the above process,

$$T_{n-2}=Q^{(n-2,n-1)}Q^{(n-2,n)} \tag{B6}$$

and

$$T_{n-1}=Q^{(n-1,n)}. \tag{B7}$$

The matrix T_{n-1} is of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & & \\ & & & & H_{n-1} & \\ 0 & 0 & \cdots & 0 & & \end{bmatrix}. \tag{B8}$$

Here H_{n-1} is a 2×2 matrix, whose elements depend on only one θ, viz., $\theta_{n(n-1)/2}$ since $Q^{(n-1,n)}$ depends solely on this angle.

Now consider the matrix Q given by

$$Q=T_1T_2\cdots T_{n-1}. \tag{B9}$$

Since the first columns of T_2 through T_{n-1} are unit vectors, the elements of the first column of Q depend only on the first

$(n-1)$ θ_i's. Since the second columns of T_3 through T_{n-1} are unit vectors, the elements of the second column of Q depend only on the first $(n-1)$ θ_i's and the next $(n-2)$ θ_i's, i.e., the first $(2n-3)$ θ_i's. Continuing the above analysis, the $(n-2)$th column of Q depends only on the first $[n(n-1)/2]-1$ θ_i's, while the penultimate and the last columns of Q depend on all the $n(n-1)/2$ θ_i's. This can be summarized by the statement that the elements of the lth column of the orthogonal matrix Q represented as in Eq. (B1) depend only on the first $l(2n-l-1)/2$ θ_i's for $l<n$ and all $n(n-1)/2$ θ_i's for $l=n$. Thus the lemma is proved.

Lemma 2. The equations for $\dot{\theta}_i$ $[i=1,2,\ldots,n(n-1)/2]$ are derived from the following set of $n(n-1)/2$ equations [cf. Eq. (2.12)]:

$$(\tilde{Q}\dot{Q})_{jk}=(\tilde{Q}\mathbf{DF}Q)_{jk}, \quad j>k. \tag{B10}$$

The above set of equations yields $(n-k)$ equations for a given value of k. These $(n-k)$ equations depend only on the first $k(2n-k-1)/2$ $\dot{\theta}_i$'s.

Proof. From the discussion following Eq. (2.12), it is clear that equations for $\dot{\theta}_i$'s are derived from Eq. (B10). Further, since $j>k$ in the above set of equations, each given value of k yields $(n-k)$ equations since all matrices have dimensions $n\times n$. The statement that these $(n-k)$ equations depend only on the first $k(2n-k-1)/2$ $\dot{\theta}_i$'s is also easily proved as follows. Note that $\dot{\theta}_i$'s appear only on the left-hand side of Eq. (B10) and

$$(\tilde{Q}\dot{Q})_{jk}=\sum_{l=1}^{n}\tilde{Q}_{jl}\dot{Q}_{lk}, \quad j=k+1,k+2,\ldots,n. \tag{B11}$$

For a given k, $\dot{\theta}_i$'s appear through the terms $\dot{Q}_{lk}$ $(l=1,2,\ldots,n)$, i.e., through the elements of the kth column of $\dot{Q}$. In the preceding lemma, we have proved that the kth column of Q depends only on the first $k(2n-k-1)/2$ θ_i's. Hence the kth column of $\dot{Q}$ depends only on the first $k(2n-k-1)/2$ $\dot{\theta}_i$'s. This completes the proof of the lemma.

As a consequence of lemma 2, equations for the first $(n-1)$ $\dot{\theta}_i$'s are given by the $(n-1)$ equations

$$\sum_{k=1}^{n}Q_{ki}\dot{Q}_{k1}=\sum_{k,j=1}^{n}Q_{ji}\mathbf{DF}_{jk}Q_{k1}, \quad i=2,3,\ldots,n. \tag{B12}$$

Only the first $(n-1)$ $\dot{\theta}_i$'s feature in the above set of equations.

The next $(n-2)$ $\dot{\theta}_i$'s, viz., $\dot{\theta}_n$, $\dot{\theta}_{n+1},\ldots,\dot{\theta}_{2n-3}$, are given by the following set of $(n-2)$ equations:

$$\sum_{k=1}^{n}Q_{ki}\dot{Q}_{k2}=\sum_{k,j=1}^{n}Q_{ji}\mathbf{DF}_{jk}Q_{k2}, \quad i=3,\ldots,n. \tag{B13}$$

Only the first $(n-1)$ and the next $(n-2)$ $\dot{\theta}_i$'s, i.e., the first $(2n-3)$ $\dot{\theta}_i$'s feature in the above set of equations.

This process is continued until we get the equation for $\dot{\theta}_{n(n-1)/2}$, which is given by

$$\sum_{k=1}^{n}Q_{kn}\dot{Q}_{k,n-1}=\sum_{k,j=1}^{n}Q_{kn}\mathbf{DF}_{kj}Q_{j,n-1}. \tag{B14}$$

All the $\dot{\theta}_i$'s feature in the above equation.

Lemma 3, The equations for $\dot{\theta}_i$'s when i is in the range $(k-1)(2n-k)/2+1$ to $k(2n-k-1)/2$ $(k=1,2,\ldots,n-1)$ depend only on the first $k(2n-k-1)/2$ θ_i's.

Proof. To make the statement of the lemma more explicit, we shall prove that the first $(n-1)$ $\dot{\theta}_i$'s, viz., $\dot{\theta}_1,\dot{\theta}_2,\ldots,\dot{\theta}_{n-1}$, depend only on the first $(n-1)$ θ_i's. The next set of $(n-2)$ $\dot{\theta}_i$'s, viz., $\dot{\theta}_n,\dot{\theta}_{n+1},\ldots,\dot{\theta}_{2n-3}$, depend only on the first $(2n-3)$ θ_i's and so on. Finally, $\dot{\theta}_{[n(n-1)/2]-2}$ and $\dot{\theta}_{[n(n-1)/2]-1}$ depend only on the first $[n(n-1)/2]-1$ θ_i's while $\dot{\theta}_{[n(n-1)/2]}$ depends on all the $n(n-1)/2$ θ_i's.

Let

$$\begin{aligned}
N&=n(n-1)/2,\\
P_1&=Q^{(1,2)},\\
P_2&=P_1Q^{(1,3)},\\
&\vdots\\
P_{n-1}&=P_{n-2}Q^{(1,n)},\\
P_n&=P_{n-1}Q^{(2,3)},\\
&\vdots\\
P_{2n-3}&=P_{2n-4}Q^{(2,n)},\\
&\vdots\\
P_{N-2}&=P_{N-3}Q^{(n-2,n-1)},\\
P_{N-1}&=P_{N-2}Q^{(n-2,n)}.
\end{aligned} \tag{B15}$$

Therefore, $Q=P_{N-1}Q^{(n-1,n)}$.

For a given value of k, we have to show that $\dot{\theta}_i$'s when i is in the range $(k-1)(2n-k)/2+1$ to $k(2n-k-1)/2$ depend only on the first $k(2n-k-1)/2$ θ_i's. For notational simplicity, we denote the starting point of the range of i values given above by $\alpha[k]$, i.e.,

$$\alpha[k]=(k-1)(2n-k)/2+1. \tag{B16}$$

The endpoint is denoted by $\omega[k]$, i.e.,

$$\omega[k]=k(2n-k-1)/2. \tag{B17}$$

We now work backwards from $k=n-1$. For $k=n-1$, $\alpha[k]=\omega[k]=n(n-1)/2=N$. Thus we have to show that $\dot{\theta}_N$ involves all θ_i's. From Eq. (B14), this is easily seen since the nth and $(n-1)$th column of Q appear in the equation. By lemma 1, these columns involve all θ_i's. Hence the statement is proved for $k=n-1$.

Next, we consider $k=n-2$. In this case, $\alpha[n-2]=N-2$ and $\omega[n-2]=N-1$. Thus, there are two $\dot{\theta}_i$'s in this range, viz., $\dot{\theta}_{\alpha[n-2]}$ and $\dot{\theta}_{\alpha[n-2]+1}$. These are given by the following set of two equations:

$$\sum_{k=1}^{n} Q_{ki}\dot{Q}_{k,n-2}=\sum_{k=1}^{n} Q_{ki}(\mathbf{DF}Q)_{k,n-2}, \quad i=n-1,n. \tag{B18}$$

Let $A=\mathbf{DF}Q$. Then the above equations imply

$$\sum_{k=1}^{n} Q_{k,n-1}\dot{Q}_{k,n-2}=\sum_{k=1}^{n} Q_{k,n-1}A_{k,n-2}, \tag{B19}$$

$$\sum_{k=1}^{n} Q_{k,n}\dot{Q}_{k,n-2}=\sum_{k=1}^{n} Q_{k,n}A_{k,n-2}, \tag{B20}$$

where

$$\begin{aligned} Q_{k,n-1}&=\sum_{j=1}^{n} (P_{N-1})_{k,j}Q_{j,n-1}^{(n-1,n)} \\ &=(P_{N-1})_{k,n-1}Q_{n-1,n-1}^{(n-1,n)}+(P_{N-1})_{k,n}Q_{n,n-1}^{(n-1,n)} \\ &=(P_{N-1})_{k,n-1}\cos\theta_N-(P_{N-1})_{k,n}\sin\theta_N. \end{aligned} \tag{B21}$$

and

$$\begin{aligned} Q_{k,n}&=\sum_{j=1}^{n} (P_{N-1})_{k,j}Q_{j,n}^{(n-1,n)}=(P_{N-1})_{k,n-1}Q_{n-1,n}^{(n-1,n)} \\ &\quad+(P_{N-1})_{k,n}Q_{n,n}^{(n-1,n)}=(P_{N-1})_{k,n-1}\sin\theta_N \\ &\quad+(P_{N-1})_{k,n}\cos\theta_N. \end{aligned} \tag{B22}$$

Multiplying Eq. (B19) by $\cos\theta_N$ and Eq. (B20) by $\sin\theta_N$ and adding the two gives

$$(P_{N-1})_{k,n-1}\dot{Q}_{k,n-2}=(P_{N-1})_{k,n-1}A_{k,n-2}. \tag{B23}$$

From Eq. (B15), we see that

$$\begin{aligned} (P_{N-1})_{k,n-1}&=\sum_{j=1}^{n} (P_{N-2})_{k,j}Q_{j,n-1}^{(n-2,n)} \\ &=(P_{N-2})_{k,n-1}, \end{aligned} \tag{B24}$$

since $Q_{n-1,n-1}^{(n-2,n)}=1$. Therefore, Eq. (B23) becomes

$$(P_{N-2})_{k,n-1}\dot{Q}_{k,n-2}=(P_{N-2})_{k,n-1}A_{k,n-2}. \tag{B25}$$

Similarly, multiplying Eq. (B19) by $\sin\theta_N$ and subtracting from Eq. (B20) multiplied by $\cos\theta_N$ gives

$$(P_{N-1})_{k,n}\dot{Q}_{k,n-2}=(P_{N-1})_{k,n}A_{k,n-2}. \tag{B26}$$

Note that $N-2=\alpha[n-2]$ and $N-1=\alpha[n-2]+1$. Therefore, to derive the expressions for $\dot{\theta}_{\alpha[n-2]}$ and $\dot{\theta}_{\alpha[n-2]+1}$, it is sufficient to solve the following two equations [cf. Eq. (B25) and Eq. (B26)]:

$$\sum_{k=1}^{n} (P_{\alpha[n-2]})_{k,n-1}\dot{Q}_{k,n-2}=\sum_{k=1}^{n} (P_{\alpha[n-2]})_{k,n-1}A_{k,n-2}, \tag{B27}$$

$$\sum_{k=1}^{n} (P_{\alpha[n-2]+1})_{k,n}\dot{Q}_{k,n-2}=\sum_{k=1}^{n} (P_{\alpha[n-2]+1})_{k,n}A_{k,n-2}.$$

We see that θ_N does not feature in the above expressions, which depend only on the first $\omega[n-2]$ θ_i's and $\dot{\theta}_i$'s.

For $k=n-3$, $\alpha[n-3]=N-5$ and $\omega[n-3]=N-3$. Thus there are three $\dot{\theta}_i$'s in this range, viz., $\dot{\theta}_{\alpha[n-3]}$, $\dot{\theta}_{\alpha[n-3]+1}$, and $\dot{\theta}_{\alpha[n-3]+2}$. These are given by solving the following set of three equations:

$$\sum_{k=1}^{n} Q_{ki}\dot{Q}_{k,n-3}=\sum_{k,j=1}^{n} Q_{ji}DF_{jk}Q_{k,n-3}, \quad i=n-2,n-1,n. \tag{B28}$$

Writing this out explicitly, we get

$$\sum_{k=1}^{n} Q_{k,n-2}\dot{Q}_{k,n-3}=\sum_{k=1}^{n} Q_{k,n-2}A_{k,n-3}, \tag{B29}$$

$$\sum_{k=1}^{n} Q_{k,n-1}\dot{Q}_{k,n-3}=\sum_{k=1}^{n} Q_{k,n-1}A_{k,n-3}, \tag{B30}$$

$$\sum_{k=1}^{n} Q_{k,n}\dot{Q}_{k,n-3}=\sum_{k=1}^{n} Q_{k,n}A_{k,n-3}. \tag{B31}$$

Only the first $\omega[n-3]=(N-3)$ $\dot{\theta}_i$'s feature in the above set of equations. We shall now show that the final expressions depend only on the first $(N-3)$ θ_i's.

Similar to the $k=n-2$ case, θ_N can be eliminated from Eq. (B30) and Eq. (B31) to give

$$\sum_{k=1}^{n} (P_{N-2})_{k,n-1}\dot{Q}_{k,n-3}=\sum_{k=1}^{n} (P_{N-2})_{k,n-1}A_{k,n-3}, \tag{B32}$$

$$\sum_{k=1}^{n} (P_{N-1})_{k,n}\dot{Q}_{k,n-3}=\sum_{k=1}^{n} (P_{N-1})_{k,n}A_{k,n-3}. \tag{B33}$$

Here $(P_{N-2})_{k,n-1}$ and $(P_{N-1})_{k,n}$ can be rewritten as

$$\begin{aligned} (P_{N-2})_{k,n-1}&=\sum_{j=1}^{n} (P_{N-3})_{k,j}Q_{j,n-1}^{(n-2,n-1)} \\ &=(P_{N-3})_{k,n-2}\sin\theta_{N-2} \\ &\quad+(P_{N-3})_{k,n-1}\cos\theta_{N-2} \end{aligned} \tag{B34}$$

and

$$\begin{aligned} (P_{N-1})_{k,n}&=\sum_{j=1}^{n} (P_{N-2})_{k,j}Q_{j,n}^{(n-2,n)} \\ &=(P_{N-2})_{k,n-2}Q_{n-2,n}^{(n-2,n)}+(P_{N-2})_{k,n}Q_{n,n}^{(n-2,n)} \\ &=(P_{N-2})_{k,n-2}\sin\theta_{N-1}+(P_{N-2})_{k,n}\cos\theta_{N-1}. \end{aligned} \tag{B35}$$

The $Q_{k,n-2}$ that appears in Eq. (B29) can be written as (making use of the fact that $Q_{n-2,n-2}^{(n-1,n)}=1$)

$$\begin{aligned}Q_{k,n-2}&=\sum_{j=1}^{n}(P_{N-1})_{k,j}Q_{j,n-2}^{(n-1,n)}\\&=(P_{N-1})_{k,n-2}\\&=(P_{N-2})_{k,n-2}Q_{n-2,n-2}^{(n-2,n)}+(P_{N-2})_{k,n}Q_{n,n-2}^{(n-2,n)}\\&=(P_{N-2})_{k,n-2}\cos\theta_{N-1}-(P_{N-2})_{k,n}\sin\theta_{N-1}.\end{aligned}\tag{B36}$$

Therefore, $\sin\theta_{N-1}\times$ Eq. (B33) $+\cos\theta_{N-1}\times$ Eq. (B29) gives

$$\sum_{k=1}^{n}(P_{N-2})_{k,n-2}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{N-2})_{k,n-2}A_{k,n-3},\tag{B37}$$

and $\cos\theta_{N-1}\times$ Eq. (B33) $-\sin\theta_{N-1}\times$ Eq. (B29) gives

$$\sum_{k=1}^{n}(P_{N-2})_{k,n}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{N-2})_{k,n}A_{k,n-3}.\tag{B38}$$

Here

$$\begin{aligned}(P_{N-2})_{k,n-2}&=\sum_{j=1}^{n}(P_{N-3})_{k,j}Q_{j,n-2}^{(n-2,n-1)}\\&=(P_{N-3})_{k,n-2}\cos\theta_{N-2}\\&\quad-(P_{N-3})_{k,n-1}\sin\theta_{N-2}.\end{aligned}\tag{B39}$$

Further, $\cos\theta_{N-2}\times$ Eq. (B37) $+\sin\theta_{N-2}\times$ Eq. (B32) gives

$$\sum_{k=1}^{n}(P_{N-3})_{k,n-2}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{N-3})_{k,n-2}A_{k,n-3}\tag{B40}$$

and $-\sin\theta_{N-2}\times$ Eq. (B37) $+\cos\theta_{N-2}\times$ Eq. (B32) gives

$$\sum_{k=1}^{n}(P_{N-3})_{k,n-1}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{N-3})_{k,n-1}A_{k,n-3}.\tag{B41}$$

Therefore, the set of three equations to solve for $\dot{\theta}_{\alpha[n-3]}$, $\dot{\theta}_{\alpha[n-3]+1}$, and $\dot{\theta}_{\alpha[n-3]+2}$ is given by

$$\sum_{k=1}^{n}(P_{N-3})_{k,n-2}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{N-3})_{k,n-2}A_{k,n-3},\tag{B42}$$

$$\sum_{k=1}^{n}(P_{N-3})_{k,n-1}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{N-3})_{k,n-1}A_{k,n-3},\tag{B43}$$

$$\sum_{k=1}^{n}(P_{N-2})_{k,n}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{N-2})_{k,n}A_{k,n-3}.\tag{B44}$$

But using the fact that $Q_{n-2,n-2}^{(n-3,n)}=Q_{n-2,n-2}^{(n-3,n-1)}=1$, we get

$$\begin{aligned}(P_{N-3})_{k,n-2}&=\sum_{j=1}^{n}(P_{N-4})_{k,j}Q_{j,n-2}^{(n-3,n)}\\&=(P_{N-4})_{k,n-2}\\&=\sum_{j=1}^{n}(P_{N-5})_{k,j}Q_{j,n-2}^{(n-3,n-1)}\\&=(P_{N-5})_{k,n-2}.\end{aligned}\tag{B45}$$

Similarly,

$$\begin{aligned}(P_{N-3})_{k,n-1}&=\sum_{j=1}^{n}(P_{N-4})_{k,j}Q_{j,n-1}^{(n-3,n)}\\&=(P_{N-4})_{k,n-1}\end{aligned}\tag{B46}$$

and

$$\begin{aligned}(P_{N-2})_{k,n}&=\sum_{j=1}^{n}(P_{N-3})_{k,j}Q_{j,n}^{(n-2,n-1)}\\&=(P_{N-3})_{k,n}.\end{aligned}\tag{B47}$$

Making use of the above simplifications and the identities

$$N-5=\alpha[n-3],\quad N-4=\alpha[n-3]+1,$$
$$N-3=\alpha[n-3]+2,\tag{B48}$$

the set of equations to be solved finally reduces to

$$\sum_{k=1}^{n}(P_{\alpha[n-3]})_{k,n-2}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{\alpha[n-3]})_{k,n-2}A_{k,n-3},\tag{B49}$$

$$\sum_{k=1}^{n}(P_{\alpha[n-3]+1})_{k,n-1}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{\alpha[n-3]+1})_{k,n-1}A_{k,n-3},\tag{B50}$$

$$\sum_{k=1}^{n}(P_{\alpha[n-3]+2})_{k,n}\dot{Q}_{k,n-3}=\sum_{k=1}^{n}(P_{\alpha[n-3]+2})_{k,n}A_{k,n-3}.\tag{B51}$$

The above set of equations depends only on the first $\omega[n-3]=(N-3)$ θ_i's and $\dot{\theta}_i$'s as we have eliminated θ_N, θ_{N-1}, and θ_{N-2} from these equations.

Similarly, for $k=n-4$, the four equations required to solve for $\dot{\theta}_{\alpha[n-4]}$, $\dot{\theta}_{\alpha[n-4]+1}$, $\dot{\theta}_{\alpha[n-4]+2}$, and $\dot{\theta}_{\alpha[n-4]+3}$ reduce to

$$\sum_{k=1}^{n}(P_{\alpha[n-4]})_{k,n-3}\dot{Q}_{k,n-4}=\sum_{k=1}^{n}(P_{\alpha[n-4]})_{k,n-3}A_{k,n-4},\tag{B52}$$

$$\sum_{k=1}^{n} (P_{\alpha[n-4]+1})_{k,n-2}\dot{Q}_{k,n-4} = \sum_{k=1}^{n} (P_{\alpha[n-4]+1})_{k,n-2}A_{k,n-4}, \tag{B53}$$

$$\sum_{k=1}^{n} (P_{\alpha[n-4]+2})_{k,n-1}\dot{Q}_{k,n-4} = \sum_{k=1}^{n} (P_{\alpha[n-4]+2})_{k,n-1}A_{k,n-4}, \tag{B54}$$

$$\sum_{k=1}^{n} (P_{\alpha[n-4]+3})_{k,n}\dot{Q}_{k,n-4} = \sum_{k=1}^{n} (P_{\alpha[n-4]+3})_{k,n}A_{k,n-4}. \tag{B55}$$

The above equations depend only on the first $\omega[n-4] = (N-6)$ θ_i's and $\dot{\theta}_i$'s.

Continuing in the same way until $k=1$, we get the final set of $n-1$ equations to solve for $\dot{\theta}_1$, $\dot{\theta}_2$, . . . , $\dot{\theta}_{n-1}$ given by

$$\sum_{k=1}^{n} (P_1)_{k,2}\dot{Q}_{k,1} = \sum_{k=1}^{n} (P_1)_{k,2}A_{k,1}, \tag{B56}$$

$$\sum_{k=1}^{n} (P_2)_{k,3}\dot{Q}_{k,1} = \sum_{k=1}^{n} (P_2)_{k,3}A_{k,1}, \tag{B57}$$

$$\vdots\ \vdots \tag{B58}$$

$$\vdots\ \vdots \tag{B59}$$

$$\sum_{k=1}^{n} (P_{n-1})_{k,n}\dot{Q}_{k,1} = \sum_{k=1}^{n} (P_{n-1})_{k,n}A_{k,1}. \tag{B60}$$

The above equations depend only on the first $\omega[1]=n-1$ θ_i's, and $\dot{\theta}_i$'s.

Therefore, we have proved that the first $(n-1)$ $\dot{\theta}_i$'s depend only on the first $(n-1)$ θ_i's, the next $n-2$ $\dot{\theta}_i$'s depend only on the first $2n-3$ θ_i's, and so on. Finally, $\dot{\theta}_{N-2}$ and $\dot{\theta}_{N-1}$ depend only on the first $(N-1)$ θ_i's, and $\dot{\theta}_N$ depends on all the $n(n-1)/2$ θ_i's. To summarize, $\dot{\theta}_{\alpha[k]}$, $\dot{\theta}_{\alpha[k]+1}$, . . . , $\dot{\theta}_{\omega[k]}$ depend only on the first $\omega[k]$ θ_i's. This completes the proof of the lemma.

We now return to the proof of the main theorem. The equation for $\dot{\lambda}_1$ is given by

$$\dot{\lambda}_1 = (\tilde{Q}\mathbf{DF}Q)_{11} \tag{B61}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} Q_{j1}\mathbf{DF}_{jk}Q_{k1}. \tag{B62}$$

The $\dot{\lambda}_1$ is seen to depend only on the first column Q_{m1} ($m=1,2,\ldots,n$) of Q. From lemma 1, the first column depends only on the first $(n-1)$ θ_i's, viz., $\theta_1,\theta_2,\ldots,\theta_{n-1}$. These θ_i's are found by using the differential equations for the corresponding $\dot{\theta}_i$'s. But, we have already proved in lemma 3 that the first $(n-1)$ $\dot{\theta}_i$'s depend only on the first $(n-1)$ θ_i's. Hence, $\dot{\lambda}_1$ also depends on the same.

The equation for $\dot{\lambda}_2$ is given by

$$\dot{\lambda}_2 = (\tilde{Q}\mathbf{DF}Q)_{22} \tag{B63}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} Q_{j2}\mathbf{DF}_{jk}Q_{k2}. \tag{B64}$$

The $\dot{\lambda}_2$ is seen to depend only on the second column Q_{m2} ($m=1,2,\ldots,n$) of Q. From lemma 1, the second column depends only on the first $(2n-3)$ θ_i's, viz., $\theta_n,\theta_{n+1},\ldots,\theta_{2n-3}$. The differential equations for the corresponding $\dot{\theta}_i$'s, used for finding these θ_i's, also depend only on the first $(2n-3)$ θ_i's (cf. lemma 3). Hence, $\dot{\lambda}_2$ also depends on the same. Similarly, $\dot{\lambda}_3$ depends only on the first $(3n-6)$ θ_i's and so on. Finally, we see that $\dot{\lambda}_{n-1}$ and $\dot{\lambda}_n$ depend on all $n(n-1)/2$ θ_i's as they depend on the last two columns of Q, which, in turn, depend on all θ_i's.

Therefore, to solve for the first Lyapunov exponent, one has to solve only n equations, i.e., $(n-1)$ equations for the first $(n-1)$ $\dot{\theta}_i$'s and the equation for $\dot{\lambda}_1$. To solve for the first two Lyapunov exponents, one has to solve $(2n-1)$ equations, i.e., $(2n-3)$ equations for the first $(2n-3)$ $\dot{\theta}_i$'s and the two equations for $\dot{\lambda}_1$ and $\dot{\lambda}_2$. Therefore, in general, to solve for the first m Lyapunov exponents, one has to solve $m(2n-m+1)/2$ equations, which is always less than $n(n+1)/2$ for $m<n$. This completes the proof of the theorem.

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