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Effect of measurement noise on Granger causality

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Most of the signals recorded in experiments are inevitably contaminated by measurement noise. Hence, it is important to understand the effect of such noise on estimating causal relations between such signals. A primary tool for estimating causality is Granger causality. Granger causality can be computed by modeling the signal using a bivariate autoregressive (AR) process. In this paper, we greatly extend the previous analysis of the effect of noise by considering a bivariate AR process of general order p. From this analysis, we analytically obtain the dependence of Granger causality on various noise-dependent system parameters. In particular, we show that measurement noise can lead to spurious Granger causality and can suppress true Granger causality. These results are verified numerically. Finally, we show how true causality can be recovered numerically using the Kalman expectation maximization algorithm.

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I. INTRODUCTION

Many experiments yield multivariate time series measurements of various phenomena. Given these multivariate time series data, it is natural to examine causal relations within these data. A popular method used to estimate such causal relations is Granger causality [1–3]. Granger causality has been recently applied to a variety of fields including neuroscience [4–20], physics [21–23], and climate change [24–27]. In this method, we assume that the recorded multichannel can be modeled as a realization of a stationary vector autoregressive (AR) process of order p [AR(p)]. We evaluate the causal relation between two time series by examining if the prediction of one series could be improved by incorporating the other. However, the experimental signals are typically noisy. Statistical analysis performed on such data may be adversely affected by the presence of noise [28]. It is therefore very important to investigate the effect of measurement noise on Granger causality estimation.

A general mathematical treatment of the effect of noise on Granger causality was given in [29]. The explicit analytical dependence of this effect on various system parameters was first derived in [30] for signals modeled by a bivariate first-order AR [AR(1)] process. Furthermore, it was shown that the adverse effect of noise on Granger causality can be mitigated by using a denoising method based on Kalman filter theory and the expectation maximization algorithm (called the KEM algorithm, in short) [30,31]. This led to further investigations on the effect of noise on Granger causality [33–36]. However, analytical expressions for the effect of measurement noise on Granger causality for time series modeled by AR(2) and higher-order AR processes were not derived in the previous work [30,31]. Since most experimental time series would need to be modeled by such higher-order processes, it is important to extend the previous analysis to bivariate AR(p) processes with measurement noise. Recently, Sommerlade *et al.* [32] investigated analytically the effects of noise on estimating Granger causality for an AR(2) process and considered a noise mitigating algorithm similar to the one in [30,31].

In this paper, we make further progress by obtaining analytical expressions that explicitly demonstrate how the measurement noise affects Granger causality (as a function of system parameters) by considering first an AR(2) process and then an AR(p) process (with unidirectional driving).

The organization of this paper is as follows: In Sec. II, we start by briefly summarizing the procedure that enables the effects of added (measurement) noise on the estimation of Granger causality to be investigated analytically. In Sec. III, we then consider a bivariate second-order autoregressive [AR(2)] process. Here, we consider two cases as follows:

Case 1: Measurement noise is added only to the driving time series $Y(t)$.

Case 2: Measurement noise is added to both time series $X(t)$ and $Y(t)$.

Explicit expressions for the effect of noise on Granger causality are derived for these two cases. In Sec. IV, similar expressions for the effect of noise on Granger causality are derived for a bivariate autoregressive process of order p [AR(p) process] with unidirectional driving (again for the two cases defined above). This analysis allows us to conclude that spurious causality can arise when noise is added to the driving time series. Furthermore, it is shown that true causality can be suppressed by the presence of noise in either time series. In Sec. V, we carry out a few numerical simulations validating the above theoretical results. In Sec. VI, we show how the noise can be removed using the KEM algorithm [30,31]. Our conclusions are given in Sec. VII.

II. THEORETICAL FRAMEWORK

We briefly outline the theoretical framework [29,30,37,38] required to compute Granger causality. Consider two time

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series $X(t)$ and $Y(t)$ modeled as a combined bivariate autoregressive process given by

$$\sum_{k=0}^{p}[a_k X(t-k)+b_k Y(t-k)]=E_1(t), \tag{1}$$

$$\sum_{k=0}^{p}[c_k X(t-k)+d_k Y(t-k)]=E_2(t). \tag{2}$$

Here a_k, b_k, c_k, and d_k are the AR coefficients and $E_i(t)$ are the temporally uncorrelated residual errors.

We rewrite the above bivariate process as two univariate processes [37] given by

$$P_1(B)X(t)=\xi(t), \quad P_2(B)Y(t)=\eta(t), \tag{3}$$

where B is the lag operator defined as $B^k X(t)=X(t-k)$ and $P_1(B)$ and $P_2(B)$ are polynomials that could have infinite order. The new noise terms $\xi(t)$ and $\eta(t)$ can now be correlated. If $\gamma_{12}(k)$ denotes the covariance at lag k between these two noises,

$$\gamma_{12}(k)\equiv \text{cov}(\xi(t),\eta(t)) \qquad k=\ldots,-1,0,1,\ldots, \tag{4}$$

then, by Pierce and Haugh's theorem [37], $Y(t)$ causes $X(t)$ in the Granger sense if and only if

$$\gamma_{12}(k)\neq 0 \quad \text{for some} \quad k>0. \tag{5}$$

Similarly $X(t)$ causes $Y(t)$ if and only if $\gamma_{12}(k)\neq 0$ for some $k<0$.

Now, consider the time series $X^{(c)}(t)$ and $Y^{(c)}(t)$ contaminated with measurement noises $\xi'(t)$ and $\eta'(t)$, respectively:

$$X^{(c)}(t)=X(t)+\xi'(t), \tag{6}$$

$$Y^{(c)}(t)=Y(t)+\eta'(t). \tag{7}$$

Here $\xi'(t)$, $\eta'(t)$ are uncorrelated Gaussian white noise processes that are uncorrelated with $X(t)$, $Y(t)$, $\xi(t)$, and $\eta(t)$. Applying $P_1(B)$ and $P_2(B)$ to $X^{(c)}(t)$ and $Y^{(c)}(t)$, respectively, and following standard procedure [29,30,38,39] we get two univariate AR processes for the noisy time series:

$$\begin{aligned} P_3^{-1}(B)P_1(B)X^{(c)}(t)&=\xi^{(c)}(t),\\ P_4^{-1}(B)P_2(B)Y^{(c)}(t)&=\eta^{(c)}(t). \end{aligned} \tag{8}$$

Here $\xi^{(c)}$ and $\eta^{(c)}$ are now uncorrelated Gaussian white noise processes. Applying the theorem of Pierce and Haugh we say that the noisy signal $Y^{(c)}(t)$ causes $X^{(c)}(t)$ in the Granger sense if and only if

$$\gamma_{12}^{(c)}(k)\equiv \text{cov}(\xi^{(c)}(t),\eta^{(c)}(t-k))\neq 0, \tag{9}$$

for some $k>0$. Similarly $X^{(c)}(t)$ cause $Y^{(c)}(t)$ if and only if

$$\gamma_{12}^{(c)}(k)\neq 0, \tag{10}$$

for some $k<0$.

This formalism can be used to show, in general terms, that spurious Granger causality can, in principle, be induced by the measurement noise [29]. Consider the following covariance generating functions (which are nothing but the z transforms of the cross covariances):

$$\begin{aligned} \Gamma_{12}(z)&=\sum_{k=-\infty}^{\infty}\gamma_{12}(k)z^k,\\ \Gamma_{12}^{(c)}(z)&=\sum_{k=-\infty}^{\infty}\gamma_{12}^{(c)}(k)z^k. \end{aligned} \tag{11}$$

These are related as [29]

$$\Gamma_{12}^{(c)}(z)=P_3^{-1}(z)P_4^{-1}(z^{-1})\Gamma_{12}(z). \tag{12}$$

Given the presence of the additional term $P_3^{-1}(z)P_4^{-1}(z^{-1})$ introduced by measurement noise, it is possible that $\gamma_{12}^{(c)}(k)\neq 0$ for some negative k even if $\gamma_{12}(k)=0$ for all $k<0$ (i.e., even if X does not cause Y). Hence, measurement noise can lead to spurious Granger causality. In the following sections, we obtain analytic expressions that demonstrate this explicitly and also obtain its dependence on system parameters.

III. A BIVARIATE AR(2) PROCESS

We now specialize the above results by considering a second-order bivariate AR(2) process given by

$$\begin{aligned} X(t)&=aX(t-1)+bY(t-1)+\xi(t),\\ Y(t)&=d_1Y(t-1)+d_2Y(t-2)+\eta(t). \end{aligned} \tag{13}$$

From the above equations, we see that Y drives X and X does not drive Y in the Granger sense of causality. But measurement noise substantially changes this situation.

Case 1: Only $Y(t)$ has measurement noise

In this case,

$$Y^{(c)}(t)=Y(t)+\eta'(t). \tag{14}$$

We rewrite the bivariate process [Eq. (13)] as two univariate processes. We proceed as follows. Equation (13) can be put in the form

$$\begin{aligned} (1-aB)X(t)&=bY(t-1)+\xi(t),\\ (1-d_1B-d_2B^2)Y(t)&=\eta(t). \end{aligned} \tag{15}$$

Let $P_2(B)=(1-d_1B-d_2B^2)$. Applying $P_2(B)$ on both sides of Eq. (14), we have

$$\begin{aligned}[1-d_1B-d_2B^2]Y^{(c)}(t)&=[1-d_1B-d_2B^2]Y(t)\\ &\quad+[1-d_1B-d_2B^2]\eta'(t).\end{aligned}$$

But from Eq. (15), $[1-d_1B-d_2B^2]Y(t)=\eta(t)$. Thus, the above expression can be rewritten as

$$[1-d_1B-d_2B^2]Y^{(c)}(t)=\eta(t)+[1-d_1B-d_2B^2]\eta'(t). \tag{16}$$

Next we rewrite $Y^{(c)}(t)$ as a univariate process. Consider the right hand side of Eq. (16). We need to find a white noise process $\eta^{(c)}(t)$ such that

$$\eta(t)+[1-d_1B-d_2B^2]\eta'(t)=(1+d_1{}'B+d_2{}'B^2)\eta^{(c)}(t). \tag{17}$$

Let

$$P_4(B) = (1 + d_1{}'B + d_2{}'B^2). \quad (18)$$

To determine $d_1{}'$, $d_2{}'$, and $\sigma^2_{\eta^{(c)}}$, we proceed as follows.

Taking variance on both sides of Eq. (17) we have

$$\sigma^2_\eta + [1 + d_1^2 + d_2^2]\sigma^2_{\eta'} = [1 + (d_1{}')^2 + (d_2{}')^2]\sigma^2_{\eta^{(c)}}. \quad (19)$$

Taking autocovariance at lag 1 on both sides of Eq. (17) we have

$$-d_1(1 - d_2)\sigma^2_{\eta'} = d_1{}'(1 + d_2{}')\sigma^2_{\eta^{(c)}}. \quad (20)$$

Finally, taking autocovariance at lag 2 on both sides of Eq. (17) we get

$$-d_2\sigma^2_{\eta'} = d_2{}'\sigma^2_{\eta^{(c)}}. \quad (21)$$

From the last equation, $d_2{}' = -d_2\frac{\sigma^2_{\eta'}}{\sigma^2_{\eta^{(c)}}}$. As $\sigma^2_{\eta^{(c)}} > \sigma^2_{\eta'}$, it follows that $|d_2{}'| < |d_2|$. Also $d_2{}'$ and d_2 are of opposite sign. From Eqs. (19), (20), and (21), we determine $d_1{}'$, $d_2{}'$, and $\sigma^2_{\eta^{(c)}}$ in terms of known quantities. Some specific solutions for the above system of equations are given in Appendix A.

Since we are considering the case where only Y has measurement noise, X is noise free. Hence, $X^{(c)}(t) = X(t)$ and $\xi^{(c)}(t) = \xi(t)$. Consequently, $P_3(B) = 1$. Substituting the expressions for P_3 and P_4 [cf. Eq. (18)] in Eq. (12), we find that the two generating functions are connected by the relation

$$\Gamma^{(c)}_{12}(z) = \left[1 + \frac{d_1{}'}{z} + \frac{d_2{}'}{z^2}\right]^{-1}\Gamma_{12}(z).$$

But,

$$\Gamma_{12}(z) = \sum_{k=0}^{\infty}\gamma_{12}(k)z^k \quad (\text{since } \gamma_{12}(k) = 0 \text{ for } k < 0),$$
$$\Gamma^{(c)}_{12}(z) = \sum_{k=-\infty}^{\infty}\gamma^{(c)}_{12}(k)z^k. \quad (22)$$

Therefore, we have

$$\Gamma^{(c)}_{12}(z) = \left\{1 - \frac{d_1{}'}{z} + \frac{1}{z^2}[(d_1{}')^2 - d_2{}'] + \frac{1}{z^3}[2d_1{}'d_2{}' - (d_1{}')^3] + \frac{1}{z^4}[(d_1{}')^2 - 3(d_1{}')^2d_2{}'] + \cdots\right\} \times [\gamma_{12}(0) + \gamma_{12}(1)z + \gamma_{12}(2)z^2 + \cdots].$$

Collecting the terms proportional to z^{-1}, z^0, z^1, etc., we have

$$\Gamma^{(c)}_{12}(z) = \cdots + \{\}z^{-3} + \{\}z^{-2} + \{-d_1{}'\gamma_{12}(0) + [(d_1{}')^2 - d_2{}']\gamma_{12}(1) + [2d_1{}'d_2{}' - (d_1{}')^3]\gamma_{12}(2) + \cdots\}z^{-1}$$
$$+ \{\gamma_{12}(0) - d_1{}'\gamma_{12}(1) + [(d_1{}')^2 - d_2{}']\gamma_{12}(2) + [2d_1{}'d_2{}' - (d_1{}')^3]\gamma_{12}(3) + \cdots\}z^0$$
$$+ \{\gamma_{12}(1) - d_1{}'\gamma_{12}(2) + [(d_1{}')^2 - d_2{}']\gamma_{12)}(3) + \cdots\}z^1 + \{\}z^2 + \{\}z^3 + \cdots.$$

From this it follows that

$$\gamma^{(c)}_{12}(-1) = -d_1{}'\gamma_{12}(0) + [(d_1{}')^2 - d_2{}']\gamma_{12}(1) + [2d_1{}'d_2{}' - (d_1{}')^3]\gamma_{12}(2) + \cdots,$$
$$\gamma^{(c)}_{12}(0) = \gamma_{12}(0) - d_1{}'\gamma_{12}(1) + [(d_1{}')^2 - d_2{}']\gamma_{12}(2) + [2d_1{}'d_2{}' - (d_1{}')^3]\gamma_{12}(3) + \cdots,$$
$$\gamma^{(c)}_{12}(1) = \gamma_{12}(1) - d_1{}'\gamma_{12}(2) + [(d_1{}')^2 - d_2{}']\gamma_{12}(3) + \cdots.$$

Here, we observe that $\gamma^{(c)}_{12}(k)$ for $k < 0$ is no longer zero when $d_1{}'$ and/or $d_2{}'$ are nonzero (that is, when measurement noise η' is nonzero). This implies that X causes $Y^{(c)}$ in the presence of noise giving rise to spurious causality. We note that the spurious causality term $\gamma^{(c)}_{12}(-1)$ is proportional to $d_1{}'$ and $d_2{}'$. This is also true for other spurious terms $\gamma^{(c)}_{12}(k)$ for $k < -1$. Hence, all spurious terms goes to zero if $d_1{}' \to 0$ and $d_2{}' \to 0$ (i.e., if Y has no added noise).

Case 2: Both $X(t)$ and $Y(t)$ have measurement noise

In this case, we add a zero mean white noise processes [$\xi'(t)$] even to $X(t)$:

$$X^{(c)}(t) = X(t) + \xi'(t). \quad (23)$$

We first rewrite $X(t)$ as a univariate process. Once this is done, we finally rewrite $X^{(c)}(t)$ as a univariate process. We proceed as follows. From Eq. (15), we have $Y(t-1) = (1 - d_1B - d_2B^2)^{-1}\eta(t-1)$ and hence

$$(1 - aB)X(t) = b(1 - d_1B - d_2B^2)^{-1}\eta(t-1) + \xi(t). \quad (24)$$

In order to rewrite $X(t)$ as a univariate process, we have to find a white noise process $\xi^{(w)}(t)$ such that

$$b(1 - d_1B - d_2B^2)^{-1}\eta(t-1) + \xi(t) = (1 - \gamma_1B - \gamma_2B^2)^{-1}\xi^{(w)}(t). \quad (25)$$

To determine γ_1, γ_2, and $\sigma^2_{\xi^{(w)}}$, we proceed as follows. Taking variance on both sides of Eq. (25) gives

$$b^2\sigma^2_\eta[1 + d_1^2 + (d_1^2 + d_2)^2 + (d_1^3 + 2d_1d_2)^2 + \cdots] + \sigma^2_\xi = [1 + \gamma_1^2 + (\gamma_1^2 + \gamma_2)^2 + (\gamma_1^3 + 2\gamma_1\gamma_2)^2 + \cdots]\sigma^2_{\xi^{(w)}}. \quad (26)$$

Taking autocovariance at lag 1 on both sides of Eq. (25) gives

$$b^2\sigma_\eta^2\big[d_1 + d_1\big(d_1^2+d_2\big) + \big(d_1^2+d_2\big)\big(d_1^3+2d_1d_2\big)+\cdots\big]$$
$$= \big[\gamma_1+\gamma_1\big(\gamma_1^2+\gamma_2\big)+\big(\gamma_1^2+\gamma_2\big)\big(\gamma_1^3+2\gamma_1\gamma_2\big)+\cdots\big]\sigma^2_{\xi^{(w)}}. \quad (27)$$

Taking autocovariance at lag 2 on both sides of Eq. (25) gives

$$b^2\sigma_\eta^2\big[\big(d_1^2+d_2\big)+d_1\big(d_1^3+2d_1d_2\big)+\cdots\big]$$
$$=\big[\big(\gamma_1^2+\gamma_2\big)+\gamma_1\big(\gamma_1^3+2\gamma_1\gamma_2\big)+\cdots\big]\sigma^2_{\xi^{(w)}}. \quad (28)$$

The expressions for γ_1, γ_2, and $\sigma^2_{\xi^{(w)}}$ in terms of system parameters are very lengthy. Some specific solutions are as follows (obtained by retaining only four terms in the approximation):

(i) If $b=0$, $d_1=0$, and $d_2=0$, then $\gamma_1=0$, $\gamma_2=0$, and $\sigma_{\xi^{(w)}}=\pm\sigma_\xi$.

(ii) If $b=1$, $d_1=0$, and $d_2=0$, then $\gamma_1=0$, $\gamma_2=0$, and $\sigma_{\xi^{(w)}}=\pm\sqrt{{\sigma_\eta}^2+{\sigma_\xi}^2}$.

We finally have

$$(1-aB)X(t)=(1-\gamma_1B-\gamma_2B^2)^{-1}\xi^{(w)}(t)$$

or

$$(1-\gamma_1B-\gamma_2B^2)(1-aB)X(t)=\xi^{(w)}(t).$$

Thus, we have rewritten $X(t)$ as a univariate process with $P_1(B)=(1-\gamma_1B-\gamma_2B^2)(1-aB)$. Applying $P_1(B)$ on both sides of Eq. (23), we get

$$P_1(B)X^{(c)}(t)=P_1(B)X(t)+P_1(B)\xi'(t).$$

That is,

$$P_1(B)X^{(c)}(t)=\xi^{(w)}(t)+P_1(B)\xi'(t).$$

Finally we are in a position to rewrite $X^{(c)}(t)$ as a univariate process. In order to accomplish this, we have to find a new white noise process $\xi^{(c)}(t)$ (of the noisy signal) such that

$$\xi^{(w)}(t)+P_1(B)\xi'(t)=P_3(B)\xi^{(c)}(t),$$

where

$$P_3(B)=(1+a_1{}'B+a_2{}'B^2+a_3{}'B^3). \quad (29)$$

Substituting for P_1 and P_3 in the equation for $\xi^{(w)}(t)$, we get

$$\xi^{(w)}(t)+[1-(a+\gamma_1B)+(\gamma_1a-\gamma_2)B^2+\gamma_2aB^3]\xi'(t)$$
$$=(1+a_1{}'B+a_2{}'B^2+a_3{}'B^3)\xi^{(c)}(t). \quad (30)$$

To determine $a_1{}'$, $a_2{}'$, $a_3{}'$, and $\sigma^2_{\xi^{(c)}}$, we proceed as follows. Taking variance on both sides of Eq. (30) gives

$$\sigma^2_{\xi^{(w)}}+\big[1+(a+\gamma_1)^2+(\gamma_1a-\gamma_2)^2+\gamma_2^2a^2\big]\sigma^2_{\xi'}$$
$$=[1+(a_1{}')^2+(a_2{}')^2+(a_3{}')^2]\sigma^2_{\xi^{(c)}}. \quad (31)$$

Taking autocovariance at lag 1 on both sides of Eq. (30) gives

$$[-(a+\gamma_1)-(a+\gamma_1)(\gamma_1a-\gamma_2)+(\gamma_1a-\gamma_2)(\gamma_2a)]\sigma^2_{\xi'}$$
$$=[a_1{}'+a_1{}'a_2{}'+a_2{}'a_3{}']\sigma^2_{\xi^{(c)}}. \quad (32)$$

Taking autocovariance at lag 2 on both sides of Eq. (30) gives

$$[(\gamma_1a-\gamma_2)-(a+\gamma_1)\gamma_2a]\sigma^2_{\xi'}=[a_2{}'+a_1{}'a_3{}']\sigma^2_{\xi^{(c)}}. \quad (33)$$

Finally, taking autocovariance at lag 3 on both sides of Eq. (30) gives

$$\gamma_2a\sigma^2_{\xi'}=a_3{}'\sigma^2_{\xi^{(c)}}. \quad (34)$$

The expressions for $a_1{}'$, $a_2{}'$, $a_3{}'$, and $\sigma^2_{\xi^{(c)}}$ are very lengthy. Few specific solutions are as given in Appendix B. It is easy to see that as $\sigma_{\xi'}\to 0$, $a_1{}'$, $a_2{}'$, and $a_3{}'\to 0$; as $\sigma_{\eta'}\to 0$, $d_1{}'$ and $d_2{}'\to 0$. In other words, all additional terms disappear in the absence of measurement noise as expected.

From the above analysis, we see that $X^{(c)}(t)$ has been rewritten in the standard form given in Eq. (8). $Y^{(c)}(t)$ was already rewritten in the standard form in the previous section. Substituting Eqs. (18) and (29) in Eq. (12), the generating function of the noisy and pure signals are related by

$$\Gamma^{(c)}_{12}(z)=(1+a_1{}'z+a_2{}'z^2+a_3{}'z^3)^{-1}$$
$$\times\left(1+\frac{d_1{}'}{z}+\frac{d_2{}'}{z^2}\right)^{-1}\Gamma_{12}(z).$$

Using Eq. (22) and expanding the terms in powers of z, we obtain

$$\begin{aligned}\Gamma^{(c)}_{12}(z)=\cdots&+\{\}z^{-2}+\{-d_1{}'\gamma_{12}(0)+[(d_1{}')^2-d_2{}']\gamma_{12}(1)+[2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(2)+\cdots\\&-a_1{}'[(d_1{}')^2-d_2{}']\gamma_{12}(0)-a_1{}'[2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(1)+\cdots+[-a_2{}'+(a_1{}')^2][2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(0)+\cdots\}z^{-1}\\&+\{\gamma_{12}(0)-d_1{}'\gamma_{12}(1)+[(d_1{}')^2-d_2{}']\gamma_{12}(2)+[2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(3)+\cdots+a_1{}'d_1{}'\gamma_{12}(0)\\&-a_1{}'[(d_1{}')^2-d_2{}']\gamma_{12}(1)-a_1{}'[2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(2)+\cdots+[-a_2{}'+(a_1{}')^2][(d_1{}')^2-d_2{}']\gamma_{12}(0)+\cdots\}z^0\\&+\{\gamma_{12}(1)-d_1{}'\gamma_{12}(2)+\cdots-a_1{}'\gamma_{12}(0)+a_1{}'d_1{}'\gamma_{12}(1)+\cdots+[-a_2{}'+(a_1{}')^2][-d_1{}']\gamma_{12}(0)+\cdots\}z^1\\&+\{\}z^2+\{\}z^3+\{\}z^4+\{\}z^5+\{\}z^6+\cdots.\end{aligned}$$

Now, collecting the terms proportional to z^{-1}, z^0, z^1, etc., the expressions for cross covariances at lag $-1,0,1$ are given by

$$\begin{aligned}\gamma^{(c)}_{12}(-1)=&-d_1{}'\gamma_{12}(0)+[(d_1{}')^2-d_2{}']\gamma_{12}(1)+[2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(2)+\cdots\\&-a_1{}'[(d_1{}')^2-d_2{}']\gamma_{12}(0)-a_1{}'[2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(1)+\cdots\\&+[-a_2{}'+(a_1{}')^2][2d_1{}'d_2{}'-(d_1{}')^3]\gamma_{12}(0)+\cdots,\end{aligned}$$

$$\begin{aligned}\gamma_{12}^{(c)}(0) &= \gamma_{12}(0) - d_1'\gamma_{12}(1) + [(d_1')^2 - d_2']\gamma_{12}(2) + [2d_1'd_2' - (d_1')^3]\gamma_{12}(3) + \cdots \\ &\quad + a_1'd_1'\gamma_{12}(0) - a_1'[(d_1')^2 - d_2']\gamma_{12}(1) - a_1'[2d_1'd_2' - (d_1')^3]\gamma_{12}(2) + \cdots \\ &\quad + [-a_2' + (a_1')^2][(d_1')^2 - d_2']\gamma_{12}(0) + \cdots, \\ \gamma_{12}^{(c)}(1) &= \gamma_{12}(1) - d_1'\gamma_{12}(2) - a_1'\gamma_{12}(0) + a_1'd_1'\gamma_{12}(1) - d_1'[-a_2' + (a_1')^2]\gamma_{12}(0) + \cdots.\end{aligned}$$

We note that $\gamma_{12}^{(c)}(-1)$, $\gamma_{12}^{(c)}(-2)$, ..., are nonzero in the presence of measurement noise leading to spurious causality. Furthermore, the spurious causality terms $\gamma_{12}^{(c)}(-1)$, $\gamma_{12}^{(c)}(-2)$, ..., are all proportional to d_1' and d_2'. Hence, they all go to zero as d_1' and $d_2' \to$ zero (i.e., if Y has no measurement noise). This happens even if a_1', a_2', and a_3' are still nonzero (i.e., even if X still has measurement noise). In other words, it is the presence of measurement noise in the driving term (Y) that causes spurious Granger causality. The true causality terms [$\gamma_{12}(k)$ for $k > 0$] are also modified by the presence of noise. For example, $\gamma_{12}(1)$ is changed to $\gamma_{12}(1) - a_1'\gamma_{12}(0) - d_1'[-a_2' + (a_1')^2]\gamma_{12}(0)$. Hence, even the true causality gets modified by the presence of noise.

The above theoretical results bring out clearly the adverse effect that noise can have on correctly determining directional influences. Numerical simulations demonstrating the effect of noise on Granger causality for an AR(2) process were already given in [30].

IV. A BIVARIATE AR(p) PROCESS

Consider a class of bivariate AR process of order p with unidirectional driving given by

$$X(t) = \sum_{i=1}^{p} a_i X(t-i) + b_l Y(t-l) + \xi(t), \tag{35}$$

$$Y(t) = \sum_{i=1}^{p} d_i Y(t-i) + \eta(t), \tag{36}$$

where $1 \leqslant l \leqslant p$. From the above equations, we see that Y drives X and X does not drive Y in the Granger sense. We now consider the effect of measurement noise on Granger causality.

Case 1: Only $Y(t)$ has measurement noise

We need to write Eqs. (35) and (36) as two univariate processes. We proceed as follows:

$$\left(1 - \sum_{i=1}^{p} a_i B^i\right) X(t) = b_l Y(t-l) + \xi(t), \tag{37}$$

$$\left(1 - \sum_{i=1}^{p} d_i B^i\right) Y(t) = \eta(t). \tag{38}$$

Let $P_2(B) = (1 - \sum_{i=1}^{p} d_i B^i)$. Applying $P_2(B)$ on both sides of Eq. (14), we get

$$\left(1 - \sum_{i=1}^{p} d_i B^i\right) Y^{(c)}(t) = \left(1 - \sum_{i=1}^{p} d_i B^i\right) Y(t) + \left(1 - \sum_{i=1}^{p} d_i B^i\right) \eta'(t).$$

From Eq. (38), $(1 - \sum_{i=1}^{p} d_i B^i) Y(t) = \eta(t)$. So, the above equation becomes

$$\left(1 - \sum_{i=1}^{p} d_i B^i\right) Y^{(c)}(t) = \eta(t) + \left(1 - \sum_{i=1}^{p} d_i B^i\right) \eta'(t). \tag{39}$$

Consider the right hand side of Eq. (39). We have to find a white noise process $\eta^{(c)}(t)$ such that

$$\eta(t) + \left(1 - \sum_{i=1}^{p} d_i B^i\right) \eta'(t) = \left(1 + \sum_{i=1}^{p} d_i' B^i\right) \eta^{(c)}(t). \tag{40}$$

Let $P_4(B) = (1 + \sum_{i=1}^{p} d_i' B^i)$. To determine $d_1', d_2', d_3', \ldots, d_p'$ and $\sigma^2_{\eta^{(c)}}$, we proceed as follows. We find the variance, autocovariance at lag 1, lag 2, ..., lag p using Eq. (40). From the resulting $p+1$ equations, we can determine d_1', d_2', d_3', ..., d_p' and $\sigma^2_{\eta^{(c)}}$ in terms of known quantities. Since X is noise free, we have $X^{(c)}(t) = X(t)$ and $\xi^{(c)}(t) = \xi(t)$. Hence, $P_3(B) = 1$. The two generating functions are connected by Eq. (12). Thus, we have

$$\Gamma_{12}^{(c)}(z) = \left[1 + \sum_{i=1}^{p} \frac{d_i'}{z^i}\right]^{-1} \Gamma_{12}(z). \tag{41}$$

The expressions for $\Gamma_{12}(z)$ and $\Gamma_{12}^{(c)}(z)$ are as given in Eq. (22). By using the above in Eq. (41) and collecting the terms proportional to z^{-1}, we get the expression for $\gamma_{12}^{(c)}(-1)$ as

$$\begin{aligned}\gamma_{12}^{(c)}(-1) &= -d_1'\gamma_{12}(0) + [(d_1')^2 - d_2']\gamma_{12}(1) \\ &\quad + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(2) \\ &\quad + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2 d_2']\gamma_{12}(3) \\ &\quad + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(4) + \cdots.\end{aligned}$$

Collecting the terms proportional to z^0 on both sides, we get

$$\begin{aligned}\gamma_{12}^{(c)}(0) &= \gamma_{12}(0) - d_1'\gamma_{12}(1) + [(d_1')^2 - d_2']\gamma_{12}(2) \\ &\quad + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(3) \\ &\quad + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2 d_2']\gamma_{12}(4) \\ &\quad + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(5) + \cdots.\end{aligned}$$

On collecting the terms proportional to z^1 on both sides, we get

$$\begin{aligned}\gamma_{12}^{(c)}(1) &= \gamma_{12}(1) - d_1'\gamma_{12}(2) + [(d_1')^2 - d_2']\gamma_{12}(3) \\ &\quad + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(4) \\ &\quad + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2 d_2']\gamma_{12}(5) \\ &\quad + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(6) + \cdots.\end{aligned}$$

We again observe that $\gamma_{12}^{(c)}(k)$ for $k < 0$ are no longer zero. This implies that X causes $Y^{(c)}$ in the presence of measurement noise giving rise to spurious Granger causality. The term $\gamma_{12}^{(c)}(-1)$ is proportional to $d_1', d_2', \ldots, d_p'$. This is also true for other spurious terms $\gamma_{12}^{(c)}(k)$ for $k < -1$. Hence, all spurious terms go to zero if $d_i' \to 0$ for $1 \leqslant i \leqslant p$, that is, if the added noise in Y goes to zero.

Case 2: Both $X(t)$ and $Y(t)$ have measurement noise

Let a zero mean white noise process $\xi'(t)$ be added even to $X(t)$ as shown in Eq. (23). We first rewrite $X(t)$ as a univariate process. We proceed as follows. From Eq. (38), we get

$$Y(t-l) = \left[1 - \sum_{i=1}^{p} d_i B^i\right]^{-1} \eta(t-l).$$

Substituting the above in Eq. (37), we have,

$$\left(1 - \sum_{i=1}^{p} a_i B^i\right) X(t) = b_l \left[1 - \sum_{i=1}^{p} d_i B^i\right]^{-1} \eta(t-l) + \xi(t). \tag{42}$$

We first find a white noise process $\xi^{(w)}(t)$ such that

$$b_l \left[1 - \sum_{i=1}^{p} d_i B^i\right]^{-1} \eta(t-l) + \xi(t) = \left[1 - \sum_{i=1}^{p} \gamma_i B^i\right]^{-1} \xi^{(w)}(t). \tag{43}$$

In order to find $\gamma_1, \gamma_2, \ldots, \gamma_p$ and $\sigma^2_{\xi^{(w)}}$, we find the variance, covariance at lag 1, lag 2, ..., lag p using Eq. (43). On solving these $p+1$ simultaneous equations, all γ_i's and $\sigma^2_{\xi^{(w)}}$ can be expressed in terms of system parameters but the corresponding expressions are very lengthy. We get [from Eqs. (42) and (43)]

$$\left(1 - \sum_{i=1}^{p} a_i B^i\right) X(t) = \left[1 - \sum_{i=1}^{p} \gamma_i B^i\right]^{-1} \xi^{(w)}(t),$$

which can be rewritten as

$$\left[1 - \sum_{i=1}^{p} \gamma_i B^i\right]\left[1 - \sum_{i=1}^{p} a_i B^i\right] X(t) = \xi^{(w)}(t).$$

Thus, we have managed to rewrite $X(t)$ as a univariate process with

$$P_1(B) = \left(1 - \sum_{i=1}^{p} \gamma_i B^i\right)\left(1 - \sum_{i=1}^{p} a_i B^i\right).$$

Now, applying $P_1(B)$ on both sides of Eq. (23), we have

$$P_1(B)X^{(c)}(t) = P_1(B)X(t) + P_1(B)\xi'(t),$$

which gives

$$P_1(B)X^{(c)}(t) = \xi^{(w)}(t) + P_1(B)\xi'(t).$$

Next we need to rewrite $X^{(c)}(t)$ in univariate form. That is, we have to find a new white noise process $\xi^{(c)}(t)$ (of the noisy signal) such that

$$\xi^{(w)}(t) + P_1(B)\xi'(t) = P_3(B)\xi^{(c)}(t), \tag{44}$$

where $P_3(B) = (1 + \sum_{i=1}^{p+1} a_i' B^i)$. In order to find $a_1', a_2', \ldots, a_{p+1}'$ and $\sigma^2_{\xi^{(c)}}$, we get a system of $(p+2)$ equations by finding variance and covariances at lag 1, lag 2, ..., lag $(p+1)$ using Eq. (44) and solving the resulting system of equations. From Eq. (12), the generating functions of the noisy and pure signals are related by

$$\Gamma_{12}^{(c)}(z) = \left[1 + \sum_{i=1}^{p+1} a_i' z^i\right]^{-1} \left[1 + \sum_{i=1}^{p} \frac{d_i'}{z^i}\right]^{-1} \Gamma_{12}(z). \tag{45}$$

By collecting the terms proportional to z^{-1}, we find cross covariance at lag -1 to be

$$\begin{aligned}
\gamma_{12}^{(c)}(-1) = {} & \{-d_1'\gamma_{12}(0) + [(d_1')^2 - d_2']\gamma_{12}(1) + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(2) \\
& + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2 d_2']\gamma_{12}(3) + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(4) + \cdots\} \\
& - a_1'\{[(d_1')^2 - d_2']\gamma_{12}(0) + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(1) \\
& + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2(d_2')]\gamma_{12}(2) + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(3) + \cdots\} \\
& + [-a_2' + (a_1')^2]\{[-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(0) + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2(d_2')]\gamma_{12}(1) \\
& + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(2) + \cdots\} \\
& + [-a_3' + 2a_1'a_2' - (a_1')^3]\{[-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2 d_2']\gamma_{12}(0) \\
& + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(1) + \cdots\} + \cdots .
\end{aligned}$$

By collecting the terms proportional to z^0, we get the expression for cross covariance at lag 0 as

$$\begin{aligned}
\gamma_{12}^{(c)}(0) = {} & \{\gamma_{12}(0) - d_1'\gamma_{12}(1) + [(d_1')^2 - d_2']\gamma_{12}(2) + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(3) \\
& + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2 d_2']\gamma_{12}(4) + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(5) + \cdots\} \\
& - a_1'\{[(d_1')^2 - d_2']\gamma_{12}(1) + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(2) \\
& + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2(d_2')]\gamma_{12}(3) + [-d_5' + 2d_1'd_4' - 3(d_1')^2 d_3']\gamma_{12}(4) + \cdots\}
\end{aligned}$$

$$+[-a_2' + (a_1')^2]\{[-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(1) + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2(d_2')]\gamma_{12}(2)$$
$$+[-d_5' + 2d_1'd_4' - 3(d_1')^2d_3']\gamma_{12}(3) + \cdots\}$$
$$+[-a_3' + 2a_1'a_2' - (a_1')^3]\{[-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2d_2']\gamma_{12}(1)$$
$$+[-d_5' + 2d_1'd_4' - 3(d_1')^2d_3']\gamma_{12}(2) + \cdots\} + \cdots .$$

Finally, on collecting the terms proportional to z^1 on both sides, we get the expression for cross covariance at lag 1 as

$$\gamma_{12}^{(c)}(1) = \{\gamma_{12}(1) - d_1'\gamma_{12}(2) + [(d_1')^2 - d_2']\gamma_{12}(3) + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(4)$$
$$+[-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2d_2']\gamma_{12}(5) + [-d_5' + 2d_1'd_4' - 3(d_1')^2d_3']\gamma_{12}(6) + \cdots\}$$
$$-a_1'\{[(d_1')^2 - d_2']\gamma_{12}(2) + [-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(3)$$
$$+[-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2(d_2')]\gamma_{12}(4) + [-d_5' + 2d_1'd_4' - 3(d_1')^2d_3']\gamma_{12}(5) + \cdots\}$$
$$+[-a_2' + (a_1')^2]\{[-d_3' + 2d_1'd_2' - (d_1')^3]\gamma_{12}(2) + [-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2(d_2')]\gamma_{12}(3)$$
$$+[-d_5' + 2d_1'd_4' - 3(d_1')^2d_3']\gamma_{12}(4) + \cdots\} + [-a_3' + 2a_1'a_2' - (a_1')^3]\{[-d_4' + 2d_1'd_3' + (d_1')^2 - 3(d_1')^2d_2']\gamma_{12}(2)$$
$$+[-d_5' + 2d_1'd_4' - 3(d_1')^2d_3']\gamma_{12}(3) + \cdots\} + \cdots .$$

We see that $\gamma_{12}^{(c)}(-1)$ is nonzero due to the terms d_1', d_2', d_3', ..., d_p'. Hence, when these terms are nonzero in the presence of measurement noise, spurious causality exists. Furthermore, since $\gamma_{12}^{(c)}(k)$ for $k > 0$ differs from $\gamma_{12}(k)$ by terms proportional to $d_1', d_2', d_3', \ldots, d_p'$, and $a_1', a_2', a_3', \ldots, a_p'$, even true causality is modified by the presence of noise in both channels.

V. SIMULATION RESULTS FOR NOISY DATA

We simulated a bivariate AR(3) process given by

$$X(t) = aX(t-1) + bY(t-1) + \xi(t),$$
$$Y(t) = d_1Y(t-1) + d_2Y(t-2) + d_3Y(t-3) + \eta(t). \quad (46)$$

The values of the parameters chosen were $a = 1$, $b = 2$, $d_1 = 0.2$, $d_2 = 0.3$, $d_3 = 0.4$, $\sigma_\xi = 0.2$, and $\sigma_\eta = 1.0$. We obtained two time series X and Y and then added Gaussian measurement noise with $\sigma_{\xi'} = 0.4$ and $\sigma_{\eta'} = 2.5$ to X and Y, respectively. The data set consisted of 100 realizations, each of length 250 ms (50 points) with the sampling period chosen as 50 ms. From this data set, Granger causality in the frequency domain (also known as the Granger causality spectrum) was estimated using the following expressions [40]:

$$I_{Y\to X}(f) = \ln \frac{S_{XX}(f)}{S_{XX}(f) - \left(\Sigma_{YY} - \frac{\Sigma_{XY}^2}{\Sigma_{XX}}\right)|H_{XY}(f)|^2}, \quad (47)$$

$$I_{X\to Y}(f) = \ln \frac{S_{YY}(f)}{S_{YY}(f) - \left(\Sigma_{XX} - \frac{\Sigma_{YX}^2}{\Sigma_{YY}}\right)|H_{YX}(f)|^2}, \quad (48)$$

where S is the spectral density matrix, H is the transfer matrix, and Σ is the noise covariance matrix. Further details can be found in [40]. Granger causality spectra $I_{X\to Y}(f)$ and $I_{Y\to X}(f)$ are plotted in Fig. 1. The true causality spectra are represented by solid lines while the computed causality spectra for the noisy data are represented by dashed lines. We observe both the presence of spurious causality and the suppression of true causality due to measurement noise.

Likewise, we simulated a bivariate AR(4) process given by

$$X(t) = aX(t-1) + bY(t-1) + \xi(t),$$
$$Y(t) = d_1Y(t-1) + d_2Y(t-2) + d_3Y(t-3) + d_4Y(t-4) + \eta(t). \quad (49)$$

The values of the parameters chosen were $a = 1$, $b = 1.5$, $d_1 = 0.1$, $d_2 = 0.2$, $d_3 = 0.3$, and $d_4 = 0.4$. The values of the standard deviation of the process noise (σ_ξ, σ_η) and added Gaussian measurement noise ($\sigma_{\xi'}$, $\sigma_{\eta'}$) were the same as for the AR(3) process. Granger causality spectra $I_{X\to Y}(f)$ and $I_{Y\to X}(f)$ are plotted in Fig. 2. Again, similar results are observed.

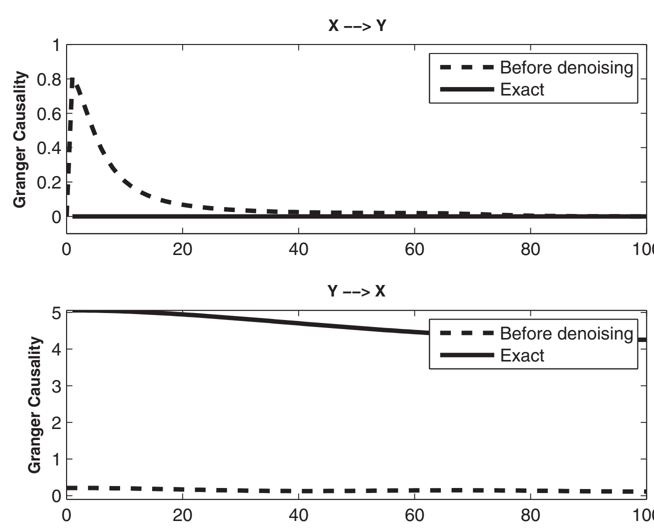

FIG. 1. Granger causality spectra for a bivariate AR(3) process: (a) causality from $X \to Y$ and (b) causality from $Y \to X$. The true causality spectra are represented by solid lines, while the spectra for noisy data are represented by dashed lines.

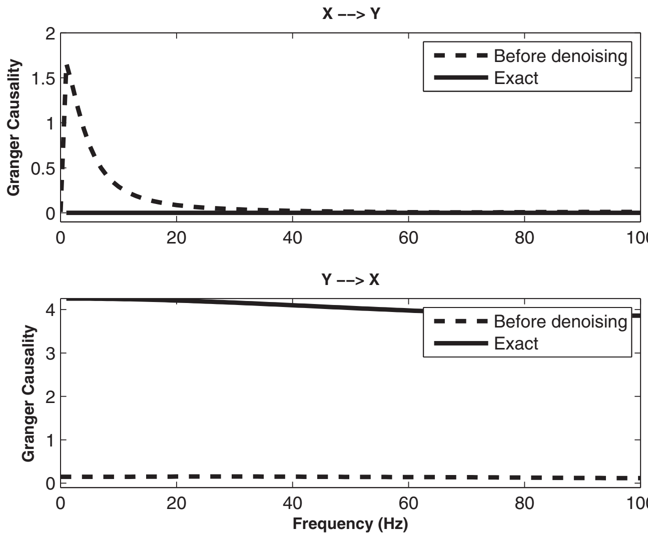

FIG. 2. Granger causality spectra for a bivariate AR(4) process: (a) causality from $X \to Y$ and (b) causality from $Y \to X$. The true causality spectra are represented by solid lines, while the spectra for noisy data are represented by dashed lines.

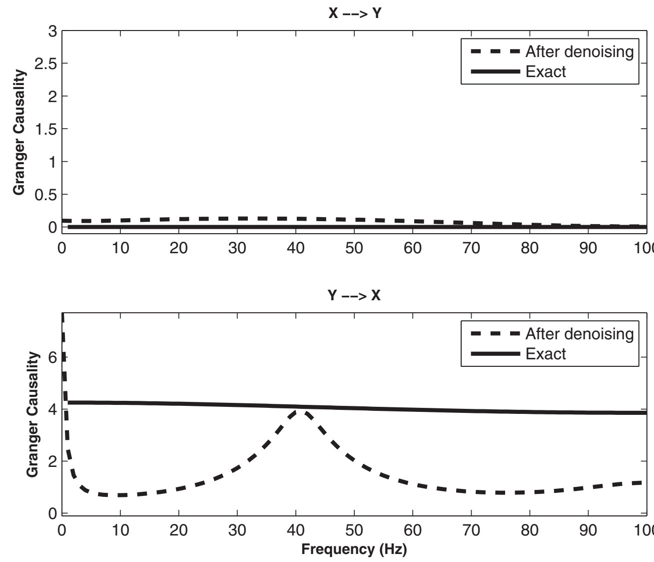

FIG. 4. Granger causality spectra for the bivariate AR(4) process in Fig. 2: (a) causality from $X \to Y$ and (b) causality from $Y \to X$. The true causality spectra are represented by solid lines, while the spectra for data denoised by the KEM algorithm are represented by dashed lines.

VI. DENOISING USING KEM ALGORITHM

The Kalman smoother in conjunction with Expectation-Maximization algorithm (also called the KEM algorithm [30,31]) was used to denoise noisy data. Further details of this algorithm can be found in [30,31].

The noisy data generated in the previous section for AR(3) and AR(4) processes were denoised by the KEM algorithm and Granger causality was again computed. The corresponding results are shown in Figs. 3 and 4 for AR(3) and AR(4) processes, respectively. We see in both cases that spurious causality is eliminated and the true causality is recovered to a great extent by the use of the KEM denoising algorithm. However, there are also artifacts like a peak at 40 Hz that was not originally present in the true causality.

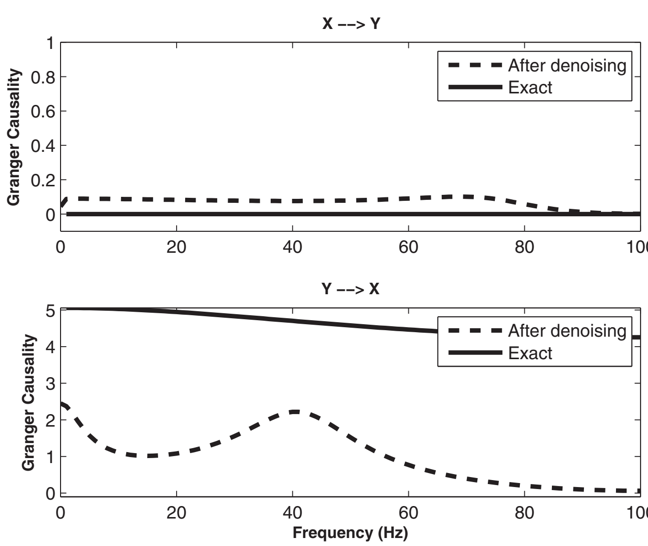

FIG. 3. Granger causality spectra for the bivariate AR(3) process in Fig. 1: (a) causality from $X \to Y$ and (b) causality from $Y \to X$. The true causality spectra are represented by solid lines, while the spectra for data denoised by the KEM algorithm are represented by dashed lines.

VII. CONCLUSIONS

We obtained analytical expressions that explicitly demonstrate the effect of measurement noise and system parameters on Granger causality estimation by considering two cases for a bivariate autoregressive process of order 2 [AR(2) process]. We showed that spurious causality can arise when noise is added to the driving time series (case 1) and true causality can be suppressed by the presence of noise in either time series (case 2). Likewise, we analytically showed similar results for a bivariate autoregressive process of order p [AR(p) process]. We demonstrated the above adverse effects of noise by numerically simulating AR(3) and AR(4) processes. Finally, using the denoising KEM algorithm we eliminated the spurious causality and recovered the true causal direction (to a substantial extent) in the above examples.

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APPENDIX A

A few specific solutions for the system of equations (19), (20), and (21) are as follows:

(i) When $d_1 = 0, d_2 = 0$, then $d_1{}' = 0, d_2{}' = 0$ and $\sigma_{\eta^{(c)}} = \pm\sqrt{{\sigma_\eta}^2 + {\sigma_{\eta'}}^2}$.

(ii) When $d_1 = 1, d_2 = 1$, we obtained the following solutions:

(a) $d_1{}' = 0$,

$$d_2{}' = -\frac{\sigma_\eta{}^2 + 3\sigma_{\eta'}{}^2 \pm \sqrt{\sigma_\eta{}^4 + 6\sigma_\eta{}^2\sigma_{\eta'}{}^2 + 5\sigma_{\eta'}{}^4}}{2\sigma_{\eta'}{}^2},$$

$$\sigma_{\eta^{(c)}} = \mp\frac{\sqrt{\sigma_\eta{}^2 + 3\sigma_{\eta'}{}^2 \mp \sqrt{\sigma_\eta{}^4 + 6\sigma_\eta{}^2\sigma_{\eta'}{}^2 + 5\sigma_{\eta'}{}^4}}}{\sqrt{2}}.$$

(b) $d_1{}' = \mp\frac{\sqrt{\sigma_\eta{}^2+\sigma_{\eta'}{}^2}}{\sigma_{\eta'}}, \quad d_2{}' = -1, \sigma_{\eta^{(c)}} = \sigma_{\eta'}$.

APPENDIX B

A few specific solutions for the system of equations (31), (32), (33), and (34) are as follows:

(i) If $a = 0$, $\gamma_1 = 0$, and $\gamma_2 = 0$ (this is true when $d_1 = 0$, $d_2 = 0$, and $b = 0$ or 1) then, $a_1' = 0$, $a_2' = 0$, $a_3' = 0$, and $\sigma_{\xi^{(c)}} = \pm\sqrt{\sigma_{\xi'}{}^2 + \sigma_{\xi^{(w)}}{}^2}$.

(ii) If $a = 1$, $\gamma_1 = 0$, and $\gamma_2 = 0$ then we get four solutions as illustrated below.

First solution: $a_1{}' = -\{\frac{2\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2[\sigma_{\xi(w)}+\sqrt{4\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2}]}{2\sigma_{\xi'}{}^2}\}$, $a_2{}' = 0$, $a_3{}' = 0$, and $\sigma_{\xi^{(c)}} = \frac{1}{2}[\sigma_{\xi^{(w)}} - \sqrt{4\sigma_{\xi'}{}^2 + \sigma_{\xi^{(w)}}{}^2}]$.

Second solution: $a_1{}' = -\{\frac{2\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2[\sigma_{\xi(w)}+\sqrt{4\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2}]}{2\sigma_{\xi'}{}^2}\}$, $a_2{}'=0$, $a_3{}'=0$, and $\sigma_{\xi^{(c)}} = \frac{1}{2}[-\sigma_{\xi^{(w)}} + \sqrt{4\sigma_{\xi'}{}^2 + \sigma_{\xi^{(w)}}{}^2}]$.

Third solution: $a_1{}' = -\{\frac{2\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2-\sigma_{\xi(w)}\sqrt{4\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2}}{2\sigma_{\xi'}{}^2}\}$, $a_2{}' = 0$, $a_3{}' = 0$, and $\sigma_{\xi^{(c)}} = \frac{1}{2}[-\sigma_{\xi^{(w)}} - \sqrt{4\sigma_{\xi'}{}^2 + \sigma_{\xi^{(w)}}{}^2}]$.

Fourth solution: $a_1{}' = -\{\frac{2\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2-\sigma_{\xi(w)}\sqrt{4\sigma_{\xi'}{}^2+\sigma_{\xi(w)}{}^2}}{2\sigma_{\xi'}{}^2}\}$, $a_2{}' = 0$, $a_3{}' = 0$, and $\sigma_{\xi^{(c)}} = \frac{1}{2}[\sigma_{\xi^{(w)}} + \sqrt{4\sigma_{\xi'}{}^2 + \sigma_{\xi^{(w)}}{}^2}]$.

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