Kick Factorization of Symplectic Maps*

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I. INTRODUCTION

There are two sanguine (perhaps almost to the point of naivete) and yet remarkably tacit assumptions made in the design and construction of accelerators and storage rings. The first, made by machine builders, is that if two machines are nearly the same, their performance (including long-term orbit stability) should be nearly the same. Without this assumption, it would be impossible to proceed since construction errors are unavoidable at some level. The second, made by accelerator theorists, is that analytical/numerical models of machine behavior, despite their approximate nature, still have relevance for real machines. Without this assumption, it would be impossible to design machines. From a mathematical perspective, what is being assumed in either case is that if two symplectic maps are *close* (in some not yet precisely defined sense), then the behavior (including long-term behavior) of systems described by these maps should be nearly the same. This note explores in a mathematical context, and for a simple example, some aspects of this assumption.

Consider a nonlinear symplectic map characterizing a nonlinear Hamiltonian system. In a Taylor series approximation, one truncates this map at a given order in phase-space variables. Such a truncated map typically produces spurious damping or growth when used to analyze the long-term behavior of trajectories. This undesirable behavior arises from the fact that the truncated map violates the symplectic condition. There are at least two ways of coping with this problem. The first consists of replacing the Taylor series with a generating function whose effect is identical to that of the Taylor series through some order (perhaps the order to which the Taylor expansion is known). The second is to replace the Taylor series by a sequence of polynomial symplectic maps that can be evaluated exactly and whose net effect is again identical to that of the Taylor series through some order. In either case, a given symplectic map is being replaced by some nearby symplectic map, and one hopes this replacement still leads to valid conclusions.

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II. A SIMPLE SYMPLECTIC MAP

Let $\mathcal{M}$ be a map acting on two dimensional phase space q, p and suppose $\mathcal{M}$ is written in the product form

$$\mathcal{M} = \mathcal{R}(\phi)\mathcal{N}. \tag{1}$$

Here $\mathcal{R}(\phi)$ denotes a linear map correspondng to rotation by angle ϕ (a simple phase advance) in the q, p plane; and $\mathcal{N}$ is a *nonlinear* map defined by the equations

$$q_f = \mathcal{N} q_i = q_i(1 - p_i)^2, \tag{2}$$

$$p_f = \mathcal{N} p_i = p_i \sum_0^{\infty} (p_i)^n = p_i/(1 - p_i). \tag{3}$$

Evidently $\mathcal{N}$ has a nonterminating Taylor expansion. It is also symplectic. Indeed, it has been selected because it has the simple Lie representation

$$\mathcal{N} = \exp : qp^2 : . \tag{4}$$

Figure 1 shows the dynamic aperture (tracking data) for $\mathcal{M}$ [1]. This figure was obtained by viewing $\mathcal{M}$ as a one-turn map, and tracking for 1000 turns the initial conditions $p_i = 0, q_i = .1, .2, \cdots .9$. Evidently, trajectories are stable if $q_i \leq .6$, and are unbounded if $q_i \geq .7$. This is a remarkably large dynamic aperture considering that, according to (3), $\mathcal{N}$ is singular at $p = 1$.

Suppose the Taylor series (3) is truncated beyond terms of degree 8. Figure 2 shows the result of tracking for the truncated map. The truncated map is not symplectic. In the figure this violation is first evident for the fourth "ring" out from the center where one sees spurious damping and on which $p \simeq .4$. The first neglected term in the Taylor series is p^9, and $(.4)^9 = 2.6 \times 10^{-4}$. In the course of 1000 turns this error could in principle accumulate to $\sim 10^3 \times 2.6 \times 10^{-4} = .26$. From the figure one sees that the accumulated error is somewhat less, but of this general order of magnitude. Consider the third ring out from the center. On this ring $p \simeq .3$ and $(.3)^9 = 2 \times 10^{-5}$. One would expect that this ring would show error effects after 10^4 turns since $10^4 \times 2 \times 10^{-5} = .2$. This has been found to be the case. We conclude that the Taylor approximation to $\mathcal{M}$ is not satisfactory for long-term tracking studies unless terms well beyond 8^{th} order are retained.

III. GENERATING FUNCTION APPROXIMATION

Suppose the map $\mathcal{N}$ is approximated by using a polynomial generating function $F(q_i, p_f)$, and the results of this generating function are required to agree with the Taylor series through 3^{rd} order [2,3]. This procedure approximates $\mathcal{N}$ by a map $\mathcal{N}_{gen}$ that is exactly symplectic. Indeed, by construction $\mathcal{N}_{gen}$ has a factored Lie product expansion of the form

$$\mathcal{N}_{gen} = \exp : qp^2 : \ \exp : f_5 : \ \exp : f_6 : \cdots, \qquad (5)$$

where the homogeneous polynomials $f_5, f_6, \cdots$ are in general not zero, but are hoped to have negligible effect for trajectories of interest.

Figure 3 shows the dynamic aperture for the map $\mathcal{R}\mathcal{N}_{gen}$. Evidently the topological features of figures 1 and 3 are similar. However, the 4^{th} ring is slightly deformed, and the rings successively farther out are successively more deformed. Indeed, there is even a somewhat ragged 7^{th} ring. These deformations are presumably the effect of the terms $f_5, f_6 \cdots$.

To study the effect of these terms, consider the map $\mathcal{N}^{1/4} = \exp : qp^2/4 :$. Approximate this map by a generating function map $\mathcal{N}'_{gen}$. For this map we have the relation

$$\mathcal{N}'_{gen} = \exp : qp^2/4 : \ \exp : f'_5 : \ \exp : f'_6 : \cdots, \qquad (6)$$

where $f'_5 \cdots$ are now at least a factor of $(1/4)^3$ smaller. This procedure allows us to construct the improved approximation

$$\mathcal{M} \simeq \mathcal{R}(\mathcal{N}'_{gen})^4. \qquad (7)$$

Figure 4 shows the dynamic aperture for the improved approximation (7). Evidently figures 1 and 4 are remarkably similar. The only noticeable difference is a slightly different nonlinear phase advance as is evident for the 5^{th} ring. This agreement is all the more remarkable when one considers that the nonlinear term $1/(1-p)$ in (3) is very significant for the outer rings.

IV. KICK APPROXIMATION

Although, as illustrated, the generating function approximation works well, it is somewhat awkward computationally since it involves the use of a Newton's method procedure, and this procedure may sometimes not converge. An alternate approach is to factor the nonlinear part of a map $\mathcal{N}$ into a product of special Lie maps called *kick* maps. Here a kick map is defined to be one for which the exponential series exp:*: terminates [3,4,5]. For the problem at hand we have constructed a kick factorized map $\mathcal{N}_{kf}$ of the form

$$\mathcal{N}_{kf} = \prod_{i=1}^{5} (\mathcal{R}_i e^{:\beta_i q^3 + \gamma_i q^4:} \mathcal{R}_i^{-1}) \qquad (8)$$

in such a way that one has the approximation

$$\mathcal{N}_{kf} = \exp : qp^2 : \exp : g_5 : \exp : g_6 : \cdots, \qquad (9)$$

where the polynomials $g_6 \cdots$ are again hoped to have negligible effect. Here the maps $\mathcal{R}_i$ are phase-space rotations whose angles, along with the coefficients β_i, γ_i, are chosen in such a way that (9) is satisfied. Kick maps have at least two advantages: They can be evaluated directly (no Newton procedure) and rapidly to machine precision; they can be inverted exactly since their inverses are also kick maps.

Figure 5 shows the dynamic aperture for the map $\mathcal{R}\mathcal{N}_{kf}$. Evidently, the first 3 rings agree well with those in figure 1. For the 4^{th} ring and beyond the agreement is not as good as the generating function approximation (figure 3). What is notable about this result is not that the kick factorization does not work well [indeed the nonlinearities associated with $1/(1-p)$ are already large on the 3^{rd} ring where the kick factorized approximation still works], but that the generating function works so remarkably well.

As in the generating function example, the effect of the terms $g_5 \cdots$ can be studied by kick factorizing $\mathcal{N}^{1/4}$ instead of $\mathcal{N}$ and then making the approximation $\mathcal{M} \simeq \mathcal{R}(\mathcal{N}'_{kf})^4$ where $\mathcal{N}'_{kf}$ is the kick factorized approximation to $\mathcal{N}^{1/4}$. Figure 6 shows the dynamic aperture for this map. Evidently, figure 6 is nearly as close to figure 1 as figure 4 is. Thus, the kick factorization approximation also works well.

V. CONCLUSIONS

Based on this brief exploration we draw the following tentative conclusions: First, map approximations that violate, even in small amount, the symplectic condition, such as truncated Taylor expansions, are not well suited to long-term tracking studies. Second, if two symplectic maps are sufficiently close, then at least gross features of their long-term behavior are in fact similar. Indeed, the symplectic approximations $(\mathcal{N}'_{gen})^4$ and $(\mathcal{N}'_{kf})^4$ probably differ more from $\mathcal{N}$ for a single turn than does the truncated Taylor expansion. Yet they give far better predictions of long-term behavior. Finally, the use of kick factorization deserves further study. Indeed, it can be shown that for the full 6-dimensional phase space symplectic maps can be approximated through 3^{rd} order using only 9 kicks, and through 11^{th} order using only 68 kicks [5]. Thus, kick factorized approximations should be ideal for realistic long-term tracking studies.

VI. FOOTNOTES AND REFERENCES

[1] For all the tracking studies of this note we have used the nonresonant phase advance $\phi = 66$ degrees.

[2] This is the method currently used in MARYLIE 3.0. For MARYLIE 5.0 f_5 and f_6 are required to vanish as well. See the MARYLIE manual and reference 3.

[3] A.J. Dragt et al., Ann. Rev. Nucl. Part. Sci. 38 (1988).

[4] J. Irwin, SSC Note 228 (1989).

[5] G. Rangarajan, Ph.D. Thesis (1990).

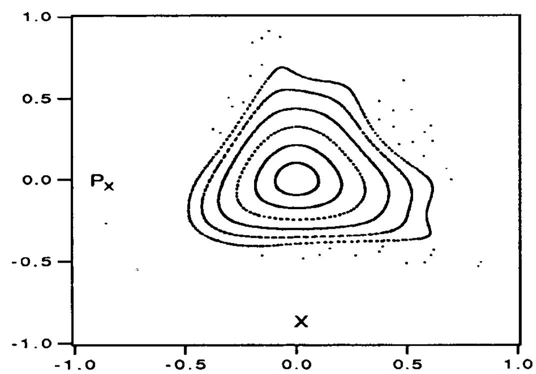

Figure 1: Tracking Results for Exact Map.

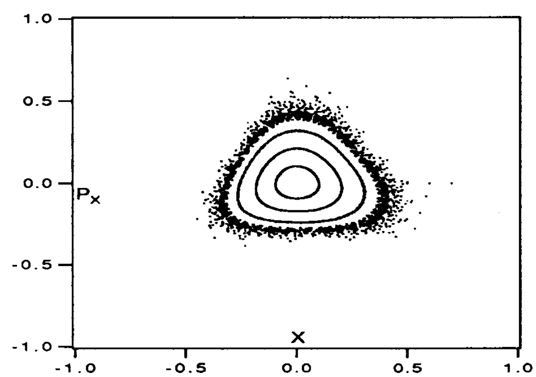

Figure 2: Truncated Taylor Approximation Results.

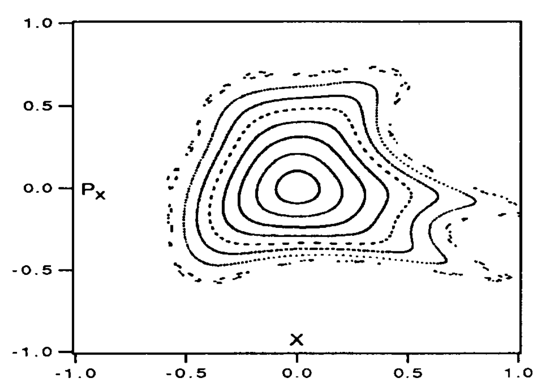

Figure 3: Generating Function Approximation Results.

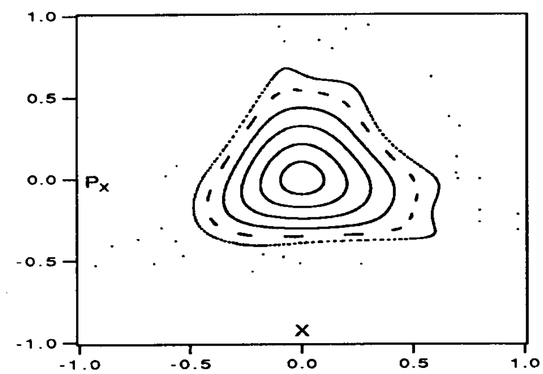

Figure 4: Improved Generating Function Approximation.

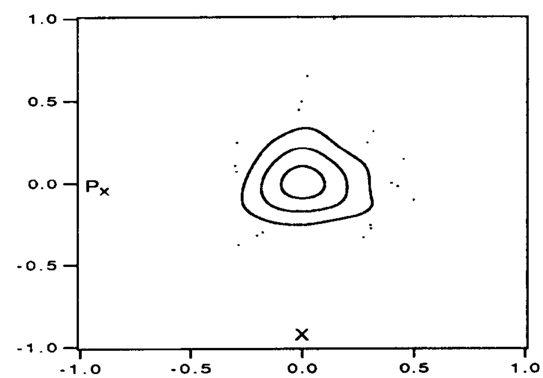

Figure 5: Kick Approximation Results.

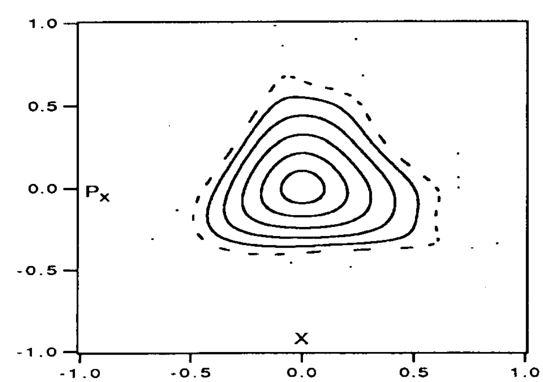

Figure 6: Improved Kick Approximation.