Synchronized Chaotic State: Stability and Pattern Formation

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Abstract—**We investigate the stability of the synchronized chaotic state for coupled maps and coupled oscillators. The stability criterion is given in terms of the coupling strengths. Pattern formation in such systems are also studied. Methods for exciting specific spatio-temporal patterns are investigated.**

Keywords—**Coupled dynamical systems, Chaotic synchronization, Generalized Turing patterns.**

I. INTRODUCTION

COUPLED dynamical systems are increasingly popular since they have applications in many areas of science – from biology[1], [2], [3] to engineering[4], [5], [6], [7], [8], [9], [10]. In this paper, we propose a general framework to analyze the stability of synchronization and pattern formation in coupled identical systems. Earlier attempts[11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29] have typically looked the two problems either at systems of very small size or at very specific coupling schemes (diffusive coupling, global all to all coupling etc. with a single coupling strength). More recently, [30], [31] introduced the notion of a master stability function that enables the analysis of general coupling topologies. However, no explicit constraints on coupling strengths themselves were given which is the goal in the present paper.

Another area of great interest in coupled systems is the study of Generalized Turing Patterns (GTP's). These differ from the classic Turing patterns [32] in the following sense. Whereas classical Turing patterns emerge from homogeneous equilibrium states, the GTP's emerge from global synchronized limit cycles or chaotic states. Moreover, the underlying coupled system need not have diffusive coupling. We show how the coupling strengths can be varied along specific paths in the parameter space to selectively realize admissible GTP's for a given system. Our methods are applicable to both coupled maps and coupled ordinary differential equations (ODEs). Commonly studied coupling schemes are used as illustrative examples.

II. STABILITY CONDITIONS

We consider the following coupled map system (the treatment for coupled oscillators is similar):

$$\mathbf{x}^i(n+1) = \mathbf{f}(\mathbf{x}^i(n)) + \frac{1}{N}\sum_{j=1}^{N} G_{ij}\cdot \mathbf{H}(\mathbf{x}^j(n)), \qquad (1)$$

where $\mathbf{x}^i(n)$ is the M-dimensional state vector of the ith map at time n and $\mathbf{H}: R^M \to R^M$ is the coupling function. We define $\mathbf{G} = [G_{ij}]$ as the coupling matrix where G_{ij} gives the coupling strength from map j to map i. The condition $\sum_{j=1}^{N} G_{ij} = 0$ is imposed to ensure that the synchronized state is a solution.

We are interested in the linear stability of the synchronized chaotic state $\mathbf{x}(n)$. The synchronized state defines the synchronization manifold in the phase space of the system. Linearizing Eq. (1) around the synchronized state, which evolves according to $\mathbf{x}(n+1) = \mathbf{f}(\mathbf{x}(n))$, we have

$$\mathbf{z}^i(n+1) = \mathbf{J}(\mathbf{x}(n))\cdot \mathbf{z}^i(n) + \frac{1}{N}\sum_{j=1}^{N} G_{ij}\cdot D\mathbf{H}(\mathbf{x}(n))\cdot \mathbf{z}^j(n), \qquad (2)$$

where $\mathbf{z}^i(n)$ denotes the ith map's deviations from $\mathbf{x}(n)$, $\mathbf{J}(\cdot)$ is the $M\times M$ Jacobian matrix for $\mathbf{f}$ and $D\mathbf{H}(\cdot)$ is the Jacobian of the coupling function $\mathbf{H}$. In terms of the $M\times N$ matrix $\mathbf{S}(n) = (\mathbf{z}^1(n)\ \mathbf{z}^2(n)\ \cdots\ \mathbf{z}^N(n))$, Eq. (2) can be recast as

$$\mathbf{S}(n+1) = \mathbf{J}(\mathbf{x}(n))\cdot \mathbf{S}(n) + \frac{1}{N} D\mathbf{H}(\mathbf{x}(n))\cdot \mathbf{S}(n)\cdot \mathbf{G}^T. \qquad (3)$$

The linear stability of Eq. (3) is determined by the eigenvalue λ of $\mathbf{G}$. Denote the corresponding eigenvector by $\mathbf{e}$ and let $\mathbf{u}(n) = \mathbf{S}(n)\mathbf{e}$ where we have suppressed the dependence on λ for notational simplicity. Then

$$\mathbf{u}(n+1) = \left(\mathbf{J}(\mathbf{x}(n)) + \frac{1}{N}\lambda\cdot D\mathbf{H}(\mathbf{x}(n))\right)\cdot \mathbf{u}(n), \quad \text{for each } \lambda. \qquad (4)$$

We note that the stability problem originally formulated in the $M\times N$ space has been reduced to a problem in a $M\times M$ space where it is often the case that $M \ll N$.

Next, we calculate the Lyapunov exponents (which depend on λ) from the above equation. If all Lyapunov exponents transverse to the synchronization manifold are negative, the synchronized state is stable since any deviation away from the synchronized manifold will quickly die down. We can formulate this in terms of the eigenvalues of G (the coupling matrix) as follows. Treat λ in Eq. (4) as a complex parameter and calculate the maximum Lyapunov exponent μ_{max} as a function of λ. This is referred to as the master stability function by Pecora and Carroll [30]. The region in the $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$ plane where $\mu_{max} < 0$ defines a stability region denoted by Ω. If the transverse eigenvalues of the coupling matrix are within Ω, then the synchronized state is stable [36]. By transverse eigenvalues we mean those eigenvalues in Eq. (4) which correspond to dynamics in the manifold transverse to the synchronization manifold. We note that, typically, Ω is obtained numerically. In some instances analytical results are possible (see below).

Stability region Ω gives constraints on the eigenvalues of the coupling matrix which ensure the stability of the synchronized state. Here we seek constraints applicable directly on the coupling strengths. This problem is dealt with by combining the master stability function with the Gershgörin disc theory. The

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Gershgörin disc theorem[37] states that all the eigenvalues of a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ are located in the union of n discs (called Gershgörin discs) where each disc is given by

$$\{z \in C : |z - a_{ii}| < \sum_{j \neq i} |a_{ji}|\}, \quad i = 1, 2, \ldots, n. \quad (5)$$

Note that $\lambda = 0$ is always an eigenvalue of $\mathbf{G}$ and its corresponding eigenvector is $(1\ 1\ \ldots 1)^T$ which is tangential to the synchronization manifold. However, for stability of the synchronized state, we only require the transverse eigenvalues to lie in Ω. Therefore, we need to remove $\lambda = 0$ before applying the Gershgörin disc theorem. In other words, for synchronized chaotic systems, the stability region does not include the origin. In order to exclude $\lambda = 0$, we appeal to an order reduction technique in matrix theory [39] which leads to a reduced $(N-1) \times (N-1)$ matrix whose eigenvalues are the same as the eigenvalues of $\mathbf{G}$ except for $\lambda = 0$.

Applying the Gershgörin theorem to the reduced matrix, the stability conditions of the synchronized dynamics can be expressed as [40]

1. $(G_{ii} - G_{ki}, 0) \in \Omega$.
2. $\sum_{j=1, j \neq i}^{N} |G_{ji} - G_{ki}| < \delta(G_{ii} - G_{ki}), \; i = 1, 2, \ldots, N; \; i \neq k$.

As k varies from 1 to N, we obtain N sets of stability conditions. Each set provides sufficient conditions constraining the coupling strengths.

As an example, we consider a coupled map system with $\mathbf{H} = \mathbf{f}$ [23], [24], [25], [26], [27], [28], [29]. Under this assumption, $D\mathbf{H} = \mathbf{J}$ and the linearized equation [cf. Eq. (4)] reduces to

$$\mathbf{u}(n+1) = (\frac{\lambda}{N} + 1)\mathbf{J}(\mathbf{x}(n))\mathbf{u}(n). \quad (6)$$

The Lyapunov exponents for Eq. (6) are easily calculated analytically. Denoting them by $\mu_1(\lambda), \mu_2(\lambda), \ldots, \mu_M(\lambda)$, we have

$$\mu_i(\lambda) = h_i + \ln|\frac{\lambda}{N} + 1|, \quad i = 1, 2, \ldots, M. \quad (7)$$

For stability, we require $\mu_{max}(\lambda) = h_{max} + \ln|\frac{\lambda}{N} + 1| < 0$ for all $\lambda \neq 0$. In other words, the stability zone is defined by

$$|\lambda + N| < N \exp(-h_{max}), \quad \lambda \neq 0. \quad (8)$$

The distance from the center of each Gershgörin disc to the boundary is easily calculated to be $\delta(G_{ii} - G_{ki}) = N \exp(-h_{max}) - |N + G_{ii} - G_{ki}|$ $(i = 1, \ldots, N, \; i \neq k)$. Thus the conditions of stability are

$$\begin{aligned} &\sum_{j=1, j \neq i}^{N} |G_{ji} - G_{ki}| + |N + (G_{ii} - G_{ki})| < \\ &N \exp(-h_{max}), \quad (9) \\ &i = 1, \ldots, N, \; i \neq k, \; k = 1 \text{ or } 2 \text{ or } \cdots \text{ or } N. \end{aligned}$$

For each k from 1 to N, we obtain a set of sufficient stability conditions.

In [41], a simple stability bound for synchronized chaos in the case of symmetric coupling was obtained as

$$[1 - \exp(-h_{max})] < G_{ij} < [1 + \exp(-h_{max})], \quad \forall i, j. \quad (10)$$

This can be derived from the general stability condition in Eq. (9) by averaging.

Next, we consider a popular system of N identical maps with P nearest neighbor coupling where exact results are available

$$\begin{aligned} \mathbf{x}_j(n+1) \;\; = \;\; & \mathbf{f}(\mathbf{x}_j(n)) + \frac{1}{2P} \sum_{p=1}^{P} a_p \left[\mathbf{f}(\mathbf{x}_{j+p}(n)) + \right. \\ & \left. \mathbf{f}(\mathbf{x}_{j-p}(n)) - 2\mathbf{f}(\mathbf{x}_j(n))\right], \end{aligned}$$

where $j = 1, 2, \ldots, N$. The coupling matrix is cyclic and shift invariant. Therefore its eigenvectors have the following form [26],

$$\mathbf{e}_l = \left(\exp(2\pi i \frac{l}{N}), \exp(4\pi i \frac{l}{N}), \cdots, \exp(2N\pi i \frac{l}{N})\right)^T, \quad (11)$$

where $l = 0, 1, \ldots, N-1$. Here $l = 0$ corresponds to the synchronized case. Eigenvalues of the coupling matrix are given by

$$\lambda_l = -\frac{2N}{P} \sum_{p=1}^{P} a_p \sin^2 \frac{\pi p(l-1)}{N}, \;\; l = 0, 1, \ldots, N-1 \quad (12)$$

Recasting inequality (8) using the above expressions for eigenvalues and their symmetry, we get the following exact stability conditions

$$|1 - \frac{2}{P} \sum_{p=1}^{P} a_p \sin^2(\frac{\pi p l}{N})| < \exp(-h_1), \quad (13)$$

where $l = 0, 1, \ldots, \frac{N}{2}$ or $\frac{N-1}{2}$.

As a numerical example we consider coupled logistic maps in the chaotic regime where $f(x) = 1 - ax^2$ with $a = 1.9$. The maximum Lyapunov exponent h_1 is 0.549. For simplicity, we restrict ourselves to $N = 5$ and $P = 2$. The stability conditions for the synchronized chaotic state are:

$$1 - \exp(-h_1) < a_1 \sin^2(\pi l/5) + a_2 \sin^2(2\pi l/5) < 1 + \exp(-h_1), \quad (14)$$

where $l = 1, 2$. On the other hand, from Eq. (9), we get the following sufficient stability bounds

$$\begin{aligned} 1 - \exp(-h_1) < \frac{1}{4}a_1 + a_2 \;\; &< \;\; 1 + \exp(-h_1), \\ 1 - \exp(-h_1) < a_1 + \frac{1}{4}a_2 \;\; &< \;\; 1 + \exp(-h_1). \end{aligned}$$

Comparing this sufficient condition to the exact solution Eq. (14), we see that our conditions are a very good approximation to the exact bound.

III. Pattern Formation

We now turn to the problem of pattern formation in coupled systems. It turns out this has an intimate connection with the stability problem we had studied in the previous sections. In the stability problem, the eigenvalues of the coupling matrix played an important role. In the study of pattern formation, the eigenvectors of the matrix play an equally important role.

Given a coupled system, using the stability bounds on coupling strengths, we selectively realize any admissible pattern we desire [46]. This is done by destabilizing a particular eigenmode. This in turn is achieved by varying the coupling strengths such that we cross the stability boundary along a particular path. Of course, to do this accurately we need exact expressions for the stability zone boundaries. However, even the sufficient conditions that we had derived earlier can provide adequate guidance in the absence of such information. We note that our approach of obtaining stability bounds in terms of the coupling strengths makes pattern selection quite simple. Since the coupled system is specified in terms of coupling strengths, varying them to achieve pattern selection is easily done.

Equally important, our approach enables us to obtain generalizations of the classic Turing patterns. In the classic approach, the synchronized state is an equilibrium point which is destabilized to give a Turing pattern with a simple time evolution of the spatial pattern. In our case, the synchronized state can be chaotic and consequently the temporal evolution of the spatial pattern is also chaotic. Further, our couplings need not be diffusive. We call the more general spatiotemporal patterns that we obtain as Generalized Turing Patterns (GTP's).

For general couplings, the spatial pattern is not necessary a Fourier mode of the linearized system like the Turing's original case. However when the coupling matrix is shift-invariant, the eigenmodes will continue to be Fourier modes. In the following we obtain an explicit strategy for adjusting the coupling parameters to get a specific pattern. The difference in the temporal evolution of the patterns that emerge from the synchronized equilibrium points and synchronized chaotic states is also highlighted.

Let us consider a system of N identical maps with P nearest neighbor coupling whose dynamical equations are given in Eq. (11). This system has a general non-diffusive coupling which is different from the diffusive coupling used in reaction-diffusion systems. However, the coupling matrix is still shift-invariant and therefore the eigenvectors of the coupling matrix shown in Eq. (11) are the Fourier modes. Further, the inequalities (13) define a stability region in the parameter space spanned by the coupling strengths a_p's. By selecting a given Fourier mode and choosing a suitable path in the parameter space we can realize the corresponding GTP. Note that, if one considers only the nearest neighbor ($P = 1$) diffusive coupling, the parameter space is one dimensional and at most two GTPs can be excited by varying the coupling strengths. By enlarging the parameter space we obtain much greater variety in terms of GTPs that can be realized.

As a numerical example we consider coupled logistic maps in the chaotic regime where $f(x) = 1 - ax^2$ with $a = 1.9$. For $N = 5$ and $P = 2$, we have the stability conditions for the synchronized chaotic state given in Eq. (14). In Fig. 1(a), we exhibit the stability region marked black in the parameter plane. Next we consider the five eigenvectors [cf. Eq. (11)] which correspond to Fourier modes in this case. The eigenvector $\mathbf{e}_0$ corresponds to the synchronized state and is excluded. Of the remaining 4 eigenvectors, only 2 are independent by symmetry of Fourier modes. We take these to be $\mathbf{e}_1$ and $\mathbf{e}_2$ [cf. Eq. (11)] corresponding to $l = 1$ and $l = 2$ respectively. We call the $l = 1$ mode the long wavelength (LW) pattern and the $l = 2$ mode the short wavelength (SW) pattern. The arrows in Figure 1a indicate paths in the parameter space which allow us to selectively destabilize one of these two modes and realize the corresponding spatial pattern.

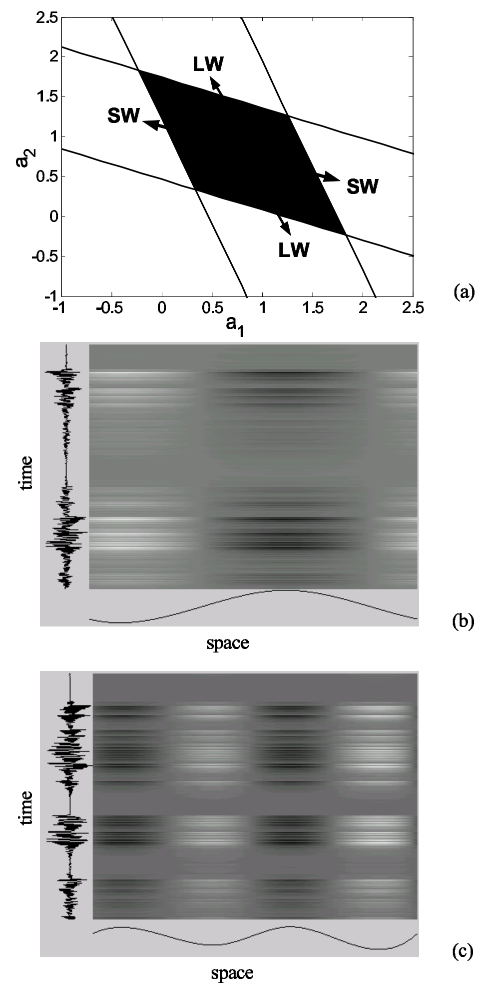

Fig. 1. Pattern selection from the synchronized chaotic state in a 1-d map lattice ($P = 2$). In (a), the region of stable synchronization (black area) and distinct pattern selection directions are shown. In (b), temporal evolution of the long wavelength pattern is given with $a_1 = 0.96, a_2 = 0.1$. In (c), temporal evolution of the short wavelength pattern with $a_1 = 0.04, a_2 = 1.1$ is given.

The main frame in Fig. 1(b) shows the temporal dynamics of the long wavelength pattern for $a_1 = 0.96$ and $a_2 = 0.1$. Here deviations from the synchronization manifold is approximated by

$$z_i(n) = x_i(n) - \sum_{j=1}^{N} \frac{x_j(n)}{N}, \ \ i = 1, 2, \cdots, N$$

with $N = 5$. To facilitate visualization, at each time step n, a continuous function is splined through the six discrete nodes: $z_1(n)$, $z_2(n)$, $\cdots$, $z_5(n)$, and $z_6(n) = z_1(n)$. Furthermore, to overcome the distortion due to the two opposite phases of a pattern, we monitor the deviation at a given node and multiply the deviations at every node by -1 whenever the deviation at the monitored node becomes negative.

Since the bifurcation undergone by the system at the boundary of the stability region is the blow-out bifurcation and there is only one attractor prior to the bifurcation, the temporal dynamics in this case is referred to as on-off intermittency [36], [26], [42], [43], [44]. The temporal evolution of the deviations at a typical node is given by the curve to the left of the main pattern frame. Its bursting behavior is characteristic of on-off intermittency. The GTP itself is given at the bottom of Fig. 1(b). For $a_1 = 0.04$ and $a_2 = 1.1$ we observe the short wavelength pattern in Fig. 1(c). The same visualization methods are used to make this figure.

IV. CONCLUSIONS

We studied stability of synchronized states in coupled identical systems using linear eigenvalue analysis. Applying Gershgörin disc theorem to the eigenvalues of the coupling matrix, quite general constraints on the coupling strengths which ensure the stability of the synchronized chaotic state were obtained. Stability of the synchronized chaotic state was studied for various examples. Then we studied pattern formation in coupled systems. By destabilizing a synchronized chaotic state, we observed the emergence of generalized Turing patterns with interesting temporal evolution. Different patterns were selectively realized in a simple manner by varying the coupling strengths along a specified path in the parameter space. In the analysis, both eigenvalues and eigenvectors of the coupling matrix were found to play crucial roles.

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