Lyapunov Exponents for Continuous-Time Dynamical Systems

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Abstract

In this article, different methods of computing Lyapunov exponents for continuous-time dynamical systems are briefly reviewed. The relative merits and demerits of these methods are pointed out.

1. Preliminaries

The problem of detecting and quantifying chaos in a wide variety of systems is an ongoing and important activity. In this context, computing the spectrum of Lyapunov exponents has proven to be the most useful dynamical diagnostic for chaotic systems.

The Lyapunov exponents give the average exponential rates of divergence or convergence of nearby orbits in the phase-space. In systems exhibiting exponential orbital divergence, small initial differences which we may not be able to resolve get magnified rapidly leading

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to loss of predictability. Any system containing atleast one positive Lyapunov exponent is defined to be chaotic, with the magnitude of the exponent reflecting the time scale on which system dynamics become unpredictable.

For systems whose equations of motions are explicitly known, there exist several methods for computing Lyapunov exponents. In this paper, we briefly describe these various methods, their advantages and disadvantages.

Let us consider an n dimensional continuous-time dynamical system,

$$\frac{dz}{dt} = F(z,t), \tag{1}$$

where $z = (z_1, z_2, ..., z_n)$ and F is a n-dimensional vector field. Let $Z(t) = z(t) - z_0(t)$ denote deviations from the fiducial trajectory $z_0(t)$. Linearizing eq(1) around $z_0(t)$, we have

$$\frac{dZ}{dt} = DF(z_0(t),t) \ . \ Z, \tag{2}$$

where DF denotes the $n \times n$ Jacobian matrix.

The linearized equations are integrated along the fiducial trajectory to yield the tangent map $M(z_0(t),t)$ which takes the set of initial variables Z^{in} into the time-evolved variables $Z(t)$, where

$$Z(t) = M(z_0(t),t) \ Z^{in}. \tag{3}$$

The evolution equation of M is given by

$$\frac{dM}{dt} = DF \ M. \tag{4}$$

Let Λ be an $n \times n$ matrix given by

$$\Lambda = \lim_{t\to\infty} (\ M \ M^t \ )^{1/2t}, \tag{5}$$

where M^t denotes the transpose of M. The Lyapunov exponents are the logarithm of the eigenvalues of Λ [1].

All the methods of computing Lyapunov exponents are either based on the QR or the singular value decomposition. In the following sections, we will describe some of these methods.

2. Singular Value Decomposition method

Let

$$M \;=\; U\; F\; V^t \tag{6}$$

be the singular value decomposition (SVD) of M into the product of the orthogonal matrices U , V and the diagonal matrix $F \;=\; \text{diag}(\; \sigma_1(t)\;,\; \sigma_2(t)\;, ...,\; \sigma_n(t)\;)$. The diagonal elements of F are called the singular values of M. The SVD is unique up to permutations of the corresponding columns, rows and diagonal elements of the matrices U , V and F . A unique decomposition can be achieved by requesting the singular value spectrum to be strictly monotonically decreasing singular value spectrum, i.e., $\sigma_1(t) > \sigma_2(t) > ... > \sigma_m(t)$. Postmultiplying eq(6) with the $M^t \;=\; V\; F\; U^t$ shows, that the squares of the singular values $\sigma_t(t)$ of M are the eigenvalues of the matrix $M\; M^t$ [2]. Therefore, from eq(5), we have the relation between the Lyapunov exponents λ_i , the eigenvalues μ_i of Λ and the singular values $\sigma_i(t)$, $i = 1, 2, ..., n$ as follows:

$$\lambda_i \;=\; \log \mu_i \;=\; \lim_{t\to\infty} \log\;(\; \sigma_i^2(t)\;)^{1/2t} = \lim_{t\to\infty} \frac{1}{t} \log \sigma_i. \tag{7}$$

The geometric intepretation of this method is explained in the reference [3].

Following Ref. [3], we will now formulate the differential equations for the quantities that are needed to compute the Lyapunov spectrum in terms of the singular value decomposition. Let us introduce a matrix E , where

$$E \;=\; \log F \;=\; \text{diag}(\; \epsilon_1,\; \epsilon_2, ..., \epsilon_n\;), \tag{8}$$

where the elements $\epsilon_i \;=\; \log \sigma_i$ $(i = 1, 2, .., n)$. Differentiating E with respect to time, yields

$$E' \;=\; F^{-1}\, F', \tag{9}$$

where

$$F' \;=\; U^t\, DF\, U\, F - U^t\, U'\, F - F\, (V')^t\, V. \tag{10}$$

This is got by subtituting eq(6) in eq(4) and differentiating w.r.t time. Due to the orthogonality of U and V , we have

$$V^t\, V' \;+\; (V')^t\, V \quad = \quad 0, \tag{11}$$

$$U^t\, U' \;+\; (U')^t\, U \quad = \quad 0. \tag{12}$$

Let us denote

$$A \quad = \quad U^t\, U', \tag{13}$$

$$B \quad = \quad -F^{-1}\, A\, F, \tag{14}$$

$$C \quad = \quad U^t\, DF\, U, \tag{15}$$

$$D \quad = \quad F^{-1}\, C\, F. \tag{16}$$

Also, $E' + (E')^t = 2E'$ yields

$$2\, E' \;=\; B \;+\; B^t \;+\; D \;+\; D^t. \tag{17}$$

To compute the Lyapunov exponents, the diagonal elements of E' need to be calculated. For this, we see from the above equation that the elements of matrices B and D are required. They are given by

$$B_{ij} \quad = \quad -A_{ij}\, \frac{\sigma_j}{\sigma_i}, \tag{18}$$

$$D_{ij} \quad = \quad C_{ij}\, \frac{\sigma_j}{\sigma_i}. \tag{19}$$

Since U is orthogonal, A is skew-symmetric and $B_{ii} = 0$, $i = 1, 2, .., n$. The diagonal elements ϵ_i' of E' therefore satisfy the equation:

$$\epsilon_i' \;=\; C_{ii}. \tag{20}$$

The above equation can used to compute the Lyapunov exponents $\lim_{t\to\infty}\epsilon_i(t)/t \; i = 1, 2.., n$ provided U is known as a function of time.

To determine $U(t)$, consider the off-diagonal elements in eq(17), the $n(n-1)/2$ equations

$$-A_{ij}\,\frac{\sigma_j}{\sigma_i} - A_{ji}\,\frac{\sigma_i}{\sigma_j} \;+\; C_{ij}\,\frac{\sigma_j}{\sigma_i} \;+\; C_{ji}\,\frac{\sigma_i}{\sigma_j} \;=\; 0, \quad i > j \tag{21}$$

To get rid of the exponentially growing quantities, eq(21) is multiplied by σ_i/σ_j . Let

$$h_{ij} \;=\; \sigma_i^2/\sigma_j^2 \;=\; \exp\left(2(\epsilon_i - \epsilon_j)\right),\; i \neq j. \tag{22}$$

Therefore, we have

$$A_{ij} \;=\; \begin{cases} \dfrac{C_{ji} \;+\; C_{ij}\; h_{ji}}{h_{ji} - 1},\; i \neq j \\ 0, \qquad\qquad\quad i = j \\ \dfrac{C_{ij} \;+\; C_{ji}\; h_{ij}}{1 - h_{ij}},\; i > j \end{cases} \tag{23}$$

The time evolution of U can now be determined by integrating the following differential equation

$$U' \;=\; U\; A. \tag{24}$$

In case of a non-degenerate spectra, the singular values constitute a strictly monotonically decreasing sequence for large time.

When the above differential equation for U is solved, the orthogonality of U is quickly lost and one has to perform reorthogonalization every now and then. In case of a degenerate Lyapunov spectra, the matrix A becomes singular. This is another disadvantage of this method. Also, it requires more operations than the QR method, which will be described in the following section. Further, evaluation of a partial Lyapunov spectrum can be computationally costly beyond a certain threshold [3].

3. QR Decomposition method

We know that any non-singular matrix can be uniquely decomposed into a product of an orthogonal matrix and an upper-triangular matrix with positive diagonal elements. Using this knowledge, we decompose the tangent map M as

$$M \;=\; Q\;R, \tag{25}$$

where Q is an $n \times n$ orthogonal matrix and R is an $n \times n$ upper-triangular matrix with positive diagonal elements R_{ii} . The Lyapunov exponents are given by

$$\lambda_i \;=\; \lim_{t\to\infty} \frac{1}{t} \log{(R_{ii})}. \tag{26}$$

In general, in the limit $t \to \infty$ the Lyapunov exponents constitute a monotonically decreasing sequence[4].

Substituting eq(25) in the eq(4), we have

$$Q'\;R \;+\; Q\;R' \;=\; DF\;Q\;R. \tag{27}$$

Premultiplying and postmultiplying the above eq with $Q^{-1} = Q^t$ and R^{-1} respectively, we have

$$Q^t\;Q' - Q^t\;DF\;Q \;=\; -\,R'\;R^{-1}. \tag{28}$$

The right hand side is an upper-triangular matrix with diagonal elements $-\,R'_{ii}/R_{ii}$, while the $Q^t\;Q'$ is a skew-symmetric matrix. Let

$$S \;=\; Q^t\;Q'. \tag{29}$$

Therefore, the differential equation for Q is given by

$$Q' \;=\; Q\;S. \tag{30}$$

The equations for the diagonal elements of R are given by

$$\frac{R'_{ii}}{R_{ii}} \;=\; (\;Q^t\;DF\;Q\;)_{ii},\;\;(1 \le i \le n). \tag{31}$$

Using the above equations, the Lyapunov exponents can be computed. This method is discussed in detail in reference[3]. This method also suffers from most of the disadvantages of the previous method.

In the following section, we shall see how things get simplified by using group-theoretical representations of the orthogonal matrix.

4. MM^t method

In this section, we describe a method utilizing representations of orthogonal matrices applied to the decompositions of the tangent map product MM^t . In this method [4], a matrix A is introduced[5], where

$$A = MM^t. \tag{32}$$

The time-evolution of A is given by the following equation:

$$\frac{dA}{dt} = DF\ A + A\ DF^t. \tag{33}$$

Since this matrix is symmetric and positive definite, it can be written as an exponential of a symmetric matrix S. Moreover, any symmetric matrix can be diagonalised by an orthogonal matrix. Therefore, we have

$$A = \exp(B) \tag{34}$$

$$= \exp(\ O\ D\ O^t) \tag{35}$$

$$= O\ \exp(D)\ O^t, \tag{36}$$

where O is an $n \times n$ orthogonal matrix, and D is an $n \times n$ diagonal matrix, whose diagonal elements are the Lyapunov exponents multiplied by time. Since D is already in the exponent, there is no need for rescaling.

An easy to obtain group-theoretical representation of the orthogonal matrix is used for the matrix O [6]. This ensures that the number of variables used to characterize the system is minimum. The number of parameters needed to characterize O and D are $n(n-1)/2$ and n respectively, giving a total of $n(n+1)/2$. This method also maintains the orthogonality without any need for rescaling. Hence, the numerical errors can never lead to loss of orthogonality.

The working of this method can be explained by taking the example of $n = 2$ case. O is represented by the following matrix:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{37}$$

D is given by

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{38}$$

The Jacobian matrix DF is given by

$$\begin{pmatrix} df_{11} & df_{12} \\ df_{21} & df_{22} \end{pmatrix}. \tag{39}$$

Substituting these expressions for A in eq(33), we have

$$\frac{d\lambda_1}{dt} = df_{11} + df_{22} + (df_{11} - df_{22})\ \cos\ 2\theta - (df_{12} + df_{21})\ \sin\ 2\theta, \tag{40}$$

$$\frac{d\lambda_2}{dt} = df_{11} + df_{22} - (df_{11} - df_{22})\ \cos\ 2\theta + (df_{12} + df_{21})\ \sin\ 2\theta. \tag{41}$$

$$\tag{42}$$

Similarly, the differential equation for θ can also be obtained. The next method to be discussed is a variant of the above method with further advantages.

5. Continuous QR method using representations of orthogonal matrices

In this method [4], the orthogonal matrix Q is represented as a product of $n(n-1)/2$ orthogonal matrices, each of which corresponds to a simple rotation in the $i-jth$ plane

$(i < j)$. Denoting the the matrix corresponding to this rotation by Q^{ij} , its matrix elements are given by:

$$Q^{(ij)}_{kl} \;=\; \begin{cases} 1 \quad \text{if } k = 1 \neq i, j; \\ \cos\theta \text{ if } k = 1 = i \text{ or} j; \\ \sin\theta \text{ if } k = i, \; l = j; \\ -\sin\theta \text{ if } k = j, \; l = i; \\ 0 \quad \text{otherwise} \end{cases}$$

where θ is an angle variable. Then, the matrix Q is represented by:

$$Q \;=\; Q^{(12)}\, Q^{(13)}...Q^{(1n)}\, Q^{(23)}...Q^{(n-1,n)}. \tag{43}$$

So, we have $n(n-1)/2$ angle variables denoted by θ_i $, i = 1, .., n(n-1)/2$. Here, Q is represented by a special orthogonal matrix because of the choice of initial conditions. We choose the identity matrix as the initial orthogonal matrix. Since we start with a matrix from the $SO(n)$ component of the group of orthogonal matrices, due to continuity, we remain in the same component for all time. Hence, we are justified in choosing Q to be an $SO(n)$ matrix. Since the upper-triangular matrix has positive diagonal elements, it can be represented as follows:

$$\begin{pmatrix} \exp\lambda_1 & r_{12} & ... & ... & r_{1n} \\ 0 & \exp\lambda_2 & r_{23} & ... & r_{2n} \\ : & : & : & : & : \\ : & : & : & : & : \\ 0 & 0 & 0 & 0 & \exp\lambda_n \end{pmatrix}. \tag{44}$$

Using the representations of Q , $Q^t\; Q'$ is given by

$$\begin{pmatrix} 0 & -f_1(\theta') & ... & -f_{n-1}(\theta') \\ f_1(\theta') & 0 & ... & -f_{2n-3}(\theta') \\ : & : & : & : \\ f_{n-1}(\theta') & ... & f_{n(n-1)/2}(\theta') & 0 \end{pmatrix}, \tag{45}$$

where $\theta' \;=\; (\theta'_1, \theta'_2, .., \theta'_{n(n-1)/2})$.

Substituting the above matrices in eq(27), we have

$$\lambda'_i \;=\; (\; Q^t\; DF\; Q\;)_{ii}. \tag{46}$$

The equations for the angles are given by

$$f_1(\theta') = (Q^t\; DF\; Q)_{21}\;;\; f_2(\theta') = (Q^t\; DF\; Q)_{31}\;;\; ...;\; f_{n(n-1)/2}(\theta') = (Q^t\; DF\; Q)_{n,n-1}.$$

The Lyapunov exponents are given by

$$\lim_{t\to\infty} \frac{\lambda_i}{t}, \quad i \ = \ 1, 2, ..., n.$$

Here again, we need minimum number of parameters to characterize the system and there is no need for rescaling. Furthermore, numerical errors can never lead to loss of orthogonality. This method has other advantages over the previous ones. The equations for θ_i are decoupled from the equations for λ_i . Hence, we need not worry about degenerate spectra. Another very interesting feature of this method is the dependence of $\lambda_1{}'$ on the first $(n-1)$ θ_i's, $\lambda_2{}'$ on the first $(2n-3)$ θ_i's and so on. Therefore, to obtain the first two λ_i 's, one needs to solve only $(2n-1)$ equations. In general, to solve for the first m Lyapunov exponents, one has to solve $m(2n-m+1)/2$ equations which is always less than $n(n+1)/2$ for $m < n$. Therefore, the partial spectrum can be easily calculated unlike in the methods listed above. This is a major advantage of this method.

In the $n = 2$ case, Q is parametrized as

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}. \tag{47}$$

R is written as,

$$\begin{pmatrix} \exp\lambda_1 & r_{12} \\ 0 & \exp\lambda_2 \end{pmatrix}. \tag{48}$$

The Jacobian matrix DF may be written as:

$$\begin{pmatrix} df_{11} & df_{12} \\ df_{21} & df_{22} \end{pmatrix}. \tag{49}$$

Substituting the above into eq(27), we have

$$\frac{d\lambda_1}{dt} \ = \ df_{11}\cos^2\theta_1 \ + \ df_{22}\sin^2\theta_1 - \frac{1}{2}\ (df_{12} + df_{21})\ \sin\ 2\theta_1, \tag{50}$$

$$\frac{d\lambda_2}{dt} \ = \ df_{11}\sin^2\theta_1 \ + \ df_{22}\cos^2\theta_1 + \frac{1}{2}\ (df_{12} + df_{21})\ \sin\ 2\theta_1. \tag{51}$$

The equation for θ_1 is given by

$$\frac{d\theta_1}{dt} \ = \ -\frac{1}{2}\ (df_{11} - df_{22})\ \sin\ 2\theta_1 \ + \ df_{12}\ \sin^2\theta_1 - df_{21}\ \cos^2\theta_1. \tag{52}$$

The above equations are numerically integrated till the desired convergence for the Lyapunov exponents λ_1/t and λ_2/t is achieved. This method also preserves the global invariances of the Lyapunov spectrum. This method is discussed in detail in the reference [4].

6. Conclusion

In this paper, we have briefly reviewed some of the methods for computing the Lyapunov exponents of continuous-time dynamical systems. The advantages accrued by using a group-theoretical representation of orthogonal matrices were brought out. It should also be noted that the methods reviewed can be applied to discrete maps with appropriate modifications [3,7].

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