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POLYNOMIAL MAP FACTORIZATION OF SYMPLECTIC MAPS

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Long-term stability studies of nonlinear Hamiltonian systems require symplectic integration algorithms which are both fast and accurate. In this paper, we study a symplectic integration method wherein the symplectic map representing the Hamiltonian system is refactorized using polynomial symplectic maps. This method is analyzed for the three degree of freedom case. Finally, we apply this algorithm to study a large particle storage ring.

Keywords: Symplectic integration; polynomial maps; Lie perturbation theory.

1. Introduction

Numerical integration algorithms are essential to study the long term single particle stability of nonlinear, nonintegrable Hamiltonian systems. However, standard numerical integration algorithms cannot be used since they are not symplectic.[1] This violation of the symplectic condition can lead to spurious chaotic or dissipative behavior. Numerical integration algorithms which satisfy the symplectic condition are called symplectic integration algorithms.[1] Several symplectic integration algorithms have been proposed in the literature.[2–21] Some of these directly use the Hamiltonian whereas others use the symplectic map[22,23] representing the nonlinear Hamiltonian system. For complicated systems like the Large Hadron Collider which has thousands of elements, using individual Hamiltonians for each element can drastically slow down the integration process. On the other hand, the map-based approach is very fast in such cases.[24,25]

One class of the map-based methods uses jolt factorization.[6,11,17,19] But there are still unanswered questions on how to best choose the underlying group and elements in the group.[26] Further, some of these methods[11,17,19] can be quite difficult to generalize to higher dimensions. Another class of methods uses solvable maps[12,21] or monomial maps.[18] Even though they are fairly straightforward to generalize to higher dimensions, they tend to introduce spurious poles and branch points not present in the original map.[26]

We investigate a new symplectic integration method where the symplectic map is refactorized using "polynomial maps" (maps whose action on phase space variables gives rise to polynomials). This does not introduce spurious poles and branch points. Moreover, it is easy to generalize to higher dimensions. Further, since it is map-based, it is also very fast. In this paper, we briefly describe the polynomial map factorization of symplectic maps with three degrees of freedom.[27] We also apply it to study a large particle storage ring.

2. Preliminaries

We restrict ourselves to three degrees of freedom nonlinear Hamiltonian system. The effect of a Hamiltonian system on a particle can be formally expressed as the action of a symplectic map $\mathcal{M}$ that takes the particle from its initial state z^{in} to its final state[22,23] $z^{f\,\mathrm{in}}$

$$z^{f\,\mathrm{in}} = \mathcal{M}\, z^{\mathrm{in}}\,, \tag{1}$$

where $z = (q_1, q_2, q_3, p_1, p_2, p_3)$. Using the Dragt–Finn factorization theorem,[22,28] the symplectic map $\mathcal{M}$ can be factorized as shown below:

$$\mathcal{M} = \hat{M} e^{:f_3:}\, e^{:f_4:} \cdots e^{:f_n:} \cdots\,. \tag{2}$$

Here $f_n(z)$ denotes a homogeneous polynomial (in z) of degree n uniquely determined by the factorization theorem.

As an application, let us consider a charged particle storage ring which typically comprises thousands of elements (drifts, quadrupoles, sextupoles, etc.). Using the above procedure, one can represent each element in the storage ring by a symplectic map. By concatenating[22] these maps together using group-theoretical methods,[29] we obtain the so-called "one-turn" map representing the entire storage ring. The one-turn map gives the final state $z^{(1)}$ of a particle after one turn around the ring as a function of its initial state $z^{(0)}$. We obtain $z^{(1)} = \mathcal{M} z^{(0)}$. To obtain the state of a particle after n turns, one has to merely iterate the above mapping N times, i.e.,

$$z^{(n)} = \mathcal{M}^n z^{(0)}\,. \tag{3}$$

Since $\mathcal{M}$ is explicitly symplectic, this gives a symplectic integration algorithm. Further, since the entire ring can be represented by a single (or at most a few) symplectic map(s), numerical integration of particle trajectories using symplectic maps is very fast.

To obtain a practical symplectic integration algorithm, we follow the perturbative approach and truncate $\mathcal{M}$ after a finite number of Lie transformations:

$$\mathcal{M} \approx \hat{M} e^{:f_3:} e^{:f_4:} \cdots e^{:f_P:}\,. \tag{4}$$

The symplectic map is said to be truncated at order P. This map is still symplectic. However, each exponential $e^{:f_n:}$ in $\mathcal{M}$ still contains an infinite number of terms in its Taylor series expansion. We get around the above problem by refactorizing $\mathcal{M}$ in terms of simpler symplectic maps which can be exactly evaluated without truncation. We use "polynomial maps" which give rise to polynomials when acting on the phase space variables. This avoids the problem of spurious poles and branch points present in generating function methods,[26] solvable map[12,21] and monomial map[18] refactorizations.

3. Symplectic Integration Using Polynomial Maps

As mentioned earlier, polynomial maps are symplectic maps which have a polynomial action on phase space variables. A simple example of a polynomial map is $\exp(: a_1 q_1^3 + a_2 p_1 :)$ where a_1, a_2 are real constants. Its explicit action on phase space variables is given by[27]

$$q_1^{f\,\mathrm{in}} = q_1^{\mathrm{in}} - a_2\,, \quad p_1^{f\,\mathrm{in}} = p_1^{\mathrm{in}} + a_1 a_2^2 - 3a_1 a_2 q_1^{\mathrm{in}} + 3a_1 (q_1^{\mathrm{in}})^2\,. \tag{5}$$

We note that the final values of the phase space variables are polynomial functions of the initial variables and therefore involve no poles or branch points. This is an example of a polynomial map.

We now determine the classes of symplectic maps which are also polynomial maps. We obtain the following simple principles which are equally applicable in higher dimensions[27]:

(1) All polynomials of the form $h(z)$ where both phase space variable and its canonically conjugate variable[30] do not occur simultaneously give rise to symplectic polynomial maps via $\exp(: h(z) :)$. We will call such $h(z)$'s polynomials of the first type.
(2) If a canonically conjugate pair q_i, p_i is present in the polynomial $h(z)$ and it appears either in the form $[a(\bar{z})q_i + g(p_i, \bar{z})]^m$ or $[a(\bar{z})p_i + g(q_i, \bar{z})]^m$ (where $m = 1, 2\ldots$, $\bar{z} = \{q_j, p_k\}$ with $j \neq k \neq i$ and a, g are polynomials in the indicated variables), then this polynomial $h(z)$ again gives rise to a symplectic polynomial map via $\exp(: h(z) :)$. If a product/sum of such factors appears in $h(z)$, each term in the product/sum is a function of different canonically conjugate pairs. We will call $h(z)$'s of the form described above as polynomials of the second type.

We now obtain a symplectic integration algorithm using polynomial maps. We restrict ourselves to symplectic maps in a six-dimensional phase space truncated at order 4. The results obtained below can be generalized to both higher orders and higher dimensions using symbolic manipulation programs. The Dragt–Finn factorization of the symplectic map is given by:

$$\mathcal{M} = \hat{M} e^{:f_3:} e^{:f_4:}\,, \tag{6}$$

where $f_3 = a_{28} q_1^3 + a_{29} q_1^2 p_1 + \cdots + a_{83} p_3^3$ and $f_4 = a_{84} q_1^4 + a_{85} q_1^3 p_1 + \cdots + a_{209} p_3^4$. Here the coefficients $a_{28}, \ldots, a_{209}$ can be explicitly computed given a Hamiltonian system[22] and are therefore known to us. The numbering of these monomial coefficients follows the standard Giorgilli scheme.[31] The above map captures the leading-order nonlinearities of the system. Since the action of the linear part $\hat{M}$ on phase space variables is well known and is already a polynomial action, we only refactorize the nonlinear part of the map using N polynomial maps.[32] This is done as follows:

$$\mathcal{M} \approx \mathcal{P} = \hat{M} e^{:h_1:} e^{:h_2:} \cdots e^{:h_N:} , \tag{7}$$

where $e^{:h_i:}$'s are symplectic polynomial maps and the numeral appearing in the subscript indexes the polynomial maps. The polynomial maps are determined by requiring that $\mathcal{P}$ agrees with $\mathcal{M}$ up to order 4, that is, when the N polynomial maps are combined, the resulting symplectic map should have all the monomials present in f_3 and f_4 with the correct coefficients up to order 4.

Using the procedure detailed in Ref. 27, it turns out that we require 23 polynomial maps for refactorization:

$$\mathcal{M} \approx \mathcal{P} = \hat{M} e^{:h_1:} e^{:h_2:} \cdots e^{:h_{23}:} . \tag{8}$$

The h_i's are given as follows:

$$\begin{aligned}
h_1 = {} & q_1^3 b_{28} + q_1^2 q_2 b_{30} + q_1^2 q_3 b_{32} + q_1 q_2^2 b_{39} + q_1 q_2 q_3 b_{41} + q_1 q_3^2 b_{46} \\
& + q_2^3 b_{64} + q_2^2 q_3 b_{66} + q_2 q_3^2 b_{71} + q_3^3 b_{80} + q_1^4 b_{84} + q_1^3 q_2 b_{86} + q_1^3 q_3 b_{88} \\
& + q_1^2 q_2^2 b_{95} + q_1^2 q_2 q_3 b_{97} + q_1^2 q_3^2 b_{102} + q_1 q_2^3 b_{120} + q_1 q_2^2 q_3 b_{122} \\
& + q_1 q_2 q_3^2 b_{127} + q_1 q_3^3 b_{136} + q_2^4 b_{175} + q_2^3 q_3 b_{177} + q_2^2 q_3^2 b_{182} \\
& + q_2 q_3^3 b_{191} + q_3^4 b_{205} , \\
h_2 = {} & [(b_{29} + b_{34}) + q_2(b_{91} + b_{106}) + p_2(b_{92} + b_{107}) + q_3(b_{93} + b_{108}) \\
& + p_3(b_{94} + b_{109})](p_1 + q_1)^3 , \\
h_3 = {} & [(-b_{29} + b_{34}) + q_2(-b_{91} + b_{106}) + p_2(-b_{92} + b_{107}) \\
& + q_3(-b_{93} + b_{108}) + p_3(-b_{94} + b_{109})](-p_1 + q_1)^3 , \\
h_4 = {} & [(b_{65} + b_{68}) + q_1(b_{121} + b_{124}) + p_1(b_{156} + b_{159}) \\
& + q_3(b_{180} + b_{186}) + p_3(b_{181} + b_{187})](p_2 + q_2)^3 , \\
h_5 = {} & [(-b_{65} + b_{68}) + q_1(-b_{121} + b_{124}) + p_1(-b_{156} + b_{159}) \\
& + q_3(-b_{180} + b_{186}) + p_3(-b_{181} + b_{187})](-p_2 + q_2)^3 , \\
h_6 = {} & [(b_{81} + b_{82}) + q_1(b_{137} + b_{138}) + p_1(b_{172} + b_{173}) + q_2(b_{192} + b_{193}) \\
& + p_2(b_{202} + b_{203})](p_3 + q_3)^3 , \\
h_7 = {} & [(-b_{81} + b_{82}) + q_1(-b_{137} + b_{138}) + p_1(-b_{172} + b_{173}) \\
& + q_2(-b_{192} + b_{193}) + p_2(-b_{202} + b_{203})](-p_3 + q_3)^3 ,
\end{aligned}$$

$$h_8 = (p_1+q_1)^2(q_2b_{35}+q_3b_{37}+q_2^2b_{110}+q_2q_3b_{112}+p_3q_2b_{113}+q_3^2b_{117}),$$

$$h_9 = (p_1+q_1)^2(p_2b_{36}+p_3b_{38}+p_2^2b_{114}+p_2q_3b_{115}+p_2p_3b_{116}+p_3^2b_{119}),$$

$$h_{10} = (p_2+q_2)^2(q_1b_{40}+q_3b_{69}+q_1^2b_{96}+q_1q_3b_{125}+p_3q_1b_{126}+q_3^2b_{188}),$$

$$h_{11} = (p_2+q_2)^2(p_1b_{55}+p_3b_{70}+p_1^2b_{146}+p_1q_3b_{160}+p_1p_3b_{161}+p_3^2b_{190}),$$

$$h_{12} = (p_3+q_3)^2(q_1b_{47}+q_2b_{72}+q_1^2b_{103}+q_1q_2b_{128}+p_2q_1b_{134}+q_2^2b_{183}),$$

$$h_{13} = (p_3+q_3)^2(p_1b_{62}+p_2b_{78}+p_1^2b_{153}+p_1q_2b_{163}+p_1p_2b_{169}+p_2^2b_{199}),$$

$$\begin{aligned} h_{14} &= p_2q_1^2b_{31}+p_3q_1^2b_{33}+p_2^2q_1b_{43}+p_2p_3q_1b_{45}+p_3^2q_1b_{48}+p_2q_1^3b_{87}\\ &\quad +p_3q_1^3b_{89}+p_2^2q_1^2b_{99}+p_2p_3q_1^2b_{101}+p_3^2q_1^2b_{104}+p_2^3q_1b_{130}\\ &\quad +p_2^2p_3q_1b_{132}+p_2p_3^2q_1b_{135}+p_3^3q_1b_{139},\end{aligned}$$

$$\begin{aligned} h_{15} &= p_1^2q_2b_{50}+p_1^2q_3b_{52}+p_1q_2^2b_{54}+p_1q_2q_3b_{56}+p_1q_3^2b_{61}+p_1^3q_2b_{141}\\ &\quad +p_1^3q_3b_{143}+p_1^2q_2^2b_{145}+p_1^2q_2q_3b_{147}+p_1^2q_3^2b_{152}+p_1q_2^3b_{155}\\ &\quad +p_1q_2^2q_3b_{157}+p_1q_2q_3^2b_{162}+p_1q_3^3b_{171},\end{aligned}$$

$$\begin{aligned} h_{16} &= p_1p_3q_2b_{57}+p_3q_2^2b_{67}+p_3^2q_2b_{73}+p_1^2p_3q_2b_{148}+p_1p_3q_2^2b_{158}\\ &\quad +p_1p_3^2q_2b_{164}+p_3q_2^3b_{178}+p_3^2q_2^2b_{184}+p_3^3q_2b_{194},\end{aligned}$$

$$\begin{aligned} h_{17} &= p_2q_1q_3b_{44}+p_2^2q_3b_{75}+p_2q_3^2b_{77}+p_2q_1^2q_3b_{100}+p_2^2q_1q_3b_{131}\\ &\quad +p_2q_1q_3^2b_{133}+p_2^3q_3b_{196}+p_2^2q_3^2b_{198}+p_2q_3^3b_{201},\end{aligned}$$

$$h_{18} = p_1p_2q_3b_{59}+p_1^2p_2q_3b_{150}+p_1p_2^2q_3b_{166}+p_1p_2q_3^2b_{168},$$

$$h_{19} = p_3q_1q_2b_{42}+p_3q_1^2q_2b_{98}+p_3q_1q_2^2b_{123}+p_3^2q_1q_2b_{129},$$

$$\begin{aligned} h_{20} &= p_1^3b_{49}+p_1^2p_2b_{51}+p_1^2p_3b_{53}+p_1p_2^2b_{58}+p_1p_2p_3b_{60}+p_1p_3^2b_{63}\\ &\quad +p_2^3b_{74}+p_2^2p_3b_{76}+p_2p_3^2b_{79}+p_3^3b_{83}+p_1^4b_{140}+p_1^3p_2b_{142}\\ &\quad +p_1^3p_3b_{144}+p_1^2p_2^2b_{149}+p_1^2p_2p_3b_{151}+p_1^2p_3^2b_{154}+p_1p_2^3b_{165}\\ &\quad +p_1p_2^2p_3b_{167}+p_1p_2p_3^2b_{170}+p_1p_3^3b_{174}+p_2^4b_{195}+p_2^3p_3b_{197}\\ &\quad +p_2^2p_3^2b_{200}+p_2p_3^3b_{204}+p_3^4b_{209},\end{aligned}$$

$$h_{21} = (p_1+q_1+p_1^2b_{105})^3+(p_2+q_2+p_2^2b_{185})^3+(p_3+q_3+p_3^2b_{208})^3,$$

$$\begin{aligned} h_{22} &= (-p_1-q_1+q_1^2b_{85})^3+(-p_2-q_2+q_2^2b_{176})^3\\ &\quad +(-p_3-q_3+q_3^2b_{206})^3,\end{aligned}$$

$$\begin{aligned} h_{23} &= (p_1+q_1)^4b_{90}+(p_1+q_1)^2(p_2+q_2)^2b_{111}+(p_1+q_1)^2(p_3+q_3)^2b_{118}\\ &\quad +(p_2+q_2)^4b_{179}+(p_2+q_2)^2(p_3+q_3)^2b_{189}+(p_3+q_3)^4b_{207}.\end{aligned}$$

By forcing the refactorized form $\mathcal{P}$ to equal the original map $\mathcal{M}$ up to order 4 and using the CBH theorem,[29] we can easily compute these unknown coefficients b_i's in terms of the known a_i's. These expressions are available from the author as part of a FORTRAN program implementing the above algorithm.

The explicit actions of the polynomial maps on phase space variables can be obtained and are given in Ref. 27. This completely determines the refactorized map $\mathcal{P}$. Each $\exp(:h_i:)$ is a polynomial map which can be evaluated exactly and is explicitly symplectic. Thus by using $\mathcal{P}$ instead of $\mathcal{M}$ in Eq. (3), we obtain an explicitly symplectic integration algorithm. Further, it is fast to evaluate and does not introduce spurious poles and branch points.

4. Applications

We have applied the method to a large particle storage ring for storing charged particles. This storage ring consists of 5109 individual elements (where these elements could be drifts, bending magnets, quadrupoles or sextupoles). If one tries to numerically integrate the trajectory of a charged particle through this ring using a conventional integration algorithm, one has to go through the ring element by element where each element is described by its own Hamiltonian. This is cumbersome and slow and further, does not respect the Hamiltonian nature of the system. On the other hand, a map-based approach where one represents the entire storage ring in terms of a single map is much faster.[24,25] When this is combined with our polynomial map refactorization, one obtains a symplectic integration algorithm which is both fast and accurate and is ideally suited for such complex real life systems. The $q_1 - p_1$ phase plot for one million turns around the ring using our polynomial map

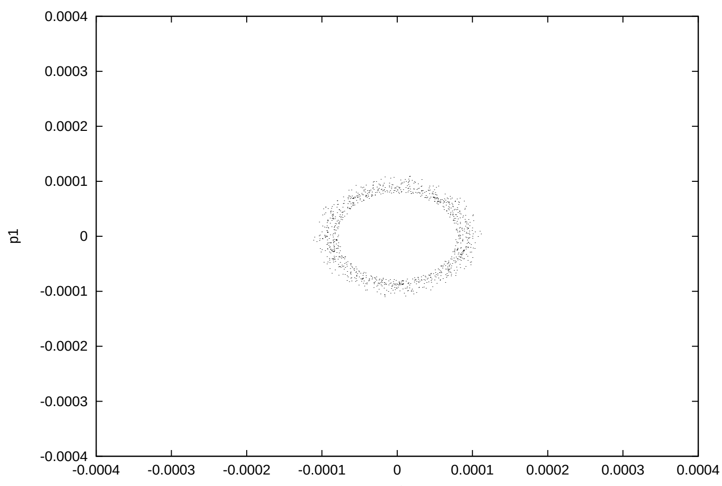

Fig. 1. The $q_1 - p_1$ phase space plot for one million turns around a storage ring using the polynomial map method (only every 1000th point is plotted).

method is given in Fig. 1. In this case, q_1 and p_1 represent the deviations from the closed orbit coordinate and momentum, respectively. From theoretical considerations, we expect the so-called betatron oscillations in these variables. This manifests itself as ellipses in the phase space plot of q_1 and p_1 variables. In Fig. 1, we observe the expected betatron oscillations. We also see the thickening of the ellipses caused by nonlinearities present in the sextupoles.

5. Conclusions

To conclude, we described a new symplectic integration algorithm based on polynomial map refactorization in three degrees of freedom. We obtained the refactorization of a given symplectic map in terms of 23 polynomial maps. This polynomial map method can be used to study long term stability of complicated nonlinear Hamiltonian systems as illustrated by our example using a large particle storage ring.

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