

Dynamics of Individual Specialization and Global Diversification in Communities

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We discuss a model of an economic community consisting of N interacting agents. The state of each agent at any time is characterized, in general, by a mixed strategy profile drawn from a space of s pure strategies. The community evolves as agents update their strategy profiles in response to payoffs received from other agents. The evolution equation is a generalization of the replicator equation. We argue that when N is sufficiently large and the payoff matrix elements satisfy suitable inequalities, the community evolves to retain the full diversity of available strategies even as individual agents specialize to pure strategies. © 1998 John Wiley & Sons, Inc.

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I. INTRODUCTION

One of the striking phenomena exhibited by a wide variety of complex adaptive systems is that individual agents or components of the system evolve to perform highly specialized tasks, and at the same time the system as a whole evolves toward a greater diversity in terms of the kinds of individual agents or components it contains or the tasks that are performed in it. Some examples of this include living systems that have evolved increasingly specialized and diverse kinds of interacting protein molecules, ecologies that develop diverse species with specialized traits, early human societies that evolve from a state where everyone shares in a small num-

ber of chores to a state with many more activities performed largely by specialists, and firms in an economic web that explore and occupy increasingly specialized and diverse niches.

In this article, we study a mathematical model of economic communities that exhibits these twin evolutionary phenomena of specialization and diversity. The system is a community of N (say, human) agents. There are s strategies or activities each agent can perform labeled by $i \in S \equiv \{1, 2, \dots, s\}$, and at the time t the agent α ($\alpha = 1, \dots, N$) performs the activity i with a probability $p_i^\alpha(t)$ (thus, $\sum_{i=1}^s p_i^\alpha(t) = 1 \forall \alpha, t$). The vector $\mathbf{p}^\alpha(t) = (p_1^\alpha(t), \dots, p_s^\alpha(t))$ is called the mixed strategy profile of agent α at time t . If $p_i^\alpha(t) = \delta_{ij}$ for some $j \in S$, then the agent α

is said to pursue the pure strategy j or to have “specialized” in the strategy j at time t . The set of vectors \mathbf{p}^α , $\alpha = 1, \dots, N$ constitutes the basic degrees of freedom of the model. The dynamics are derived by the equation

$$\dot{p}_i^\alpha(t) = p_i^\alpha(t) \left[\sum_{\beta \neq \alpha} \sum_j a_{ij} p_j^\beta(t) - \sum_{\beta \neq \alpha} \sum_{k,j} p_k^\beta(t) a_{kj} p_j^\beta(t) \right], \quad (1.1)$$

$$1 \leq \alpha \leq N, 1 \leq i \leq s,$$

which determines the rate of change of an individual \mathbf{p}^α in terms of the current mixed strategy profiles of all the agents and the payoff matrix $A = [[a_{ij}]]$.

We motivate this model as follows: Agents interact with each other on a short time scale, receive payoffs based on each other’s activity, and update their individual strategy profiles on a longer time scale so as to increase their payoffs. As is usual in game theory, a_{ij} denotes the payoff received by an agent pursuing a pure strategy i in a single interaction with an agent pursuing the pure strategy j . Then the average payoff received by the agent α from the rest of the community in the period t to $t + \Delta t$ is proportional to $\Delta t \sum_{\beta \neq \alpha} \sum_{k,j} p_k^\beta(t) a_{kj} p_j^\beta(t)$. This assumes that every agent interacts equally often with all other agents and that there is a separation of time scales: Δt can be chosen long enough for there to be a statistically sufficient number of interactions during the period, yet short enough that the change in the strategy profiles during this period can be ignored in the computation of the average payoff.

If α had played the pure strategy i in this period, she would have received an average payoff proportional to $\Delta t \sum_{\beta \neq \alpha} \sum_j a_{ij} p_j^\beta(t)$. The agent α increments p_i^α by an amount proportional to p_i^α as well as to the difference between the average payoff she would have gotten in this interval if she had pursued a pure strategy i and the average payoff she actually received: $\Delta p_i^\alpha = c \Delta t p_i^\alpha \left[\sum_{\beta \neq \alpha} \sum_j a_{ij} p_j^\beta(t) - \sum_{\beta \neq \alpha} \sum_{k,j} p_k^\beta(t) a_{kj} p_j^\beta(t) \right]$, where c is a constant. Equation (1.1) follows upon dividing by Δt , taking the limit, and rescaling time by a factor c . By construction, each agent makes a positive change in the weight of strategy i in her own strategy profile if she perceives that the pure strategy i would give a higher payoff in the current environment than her current strategy profile, and a negative change if it were to give a lower payoff.

Eq. (1.1) is nothing but the “multipopulation replicator equation” discussed in [1] (and references therein). There each α represents a population, and p_i^α represents the fraction of individuals in the population α pursuing the strategy i . Since for us each α represents an individual and not a population, we refer to dynamics specified by (1.1) as simply the generalized replicator dynamic (GRD). By contrast the replicator dynamics (which we hereafter refer to as the pure replicator dynamics (PRD)) are given by (see [2])

$$\dot{x}_i(t) = x_i(t) \left[\sum_j a_{ij} x_j(t) - \sum_{k,j} x_k(t) a_{kj} x_j(t) \right], \quad i = 1, \dots, s. \quad (1.2)$$

This is a standard model in evolutionary biology describing the growth and decay of s species under selection pressure with x_i representing the fraction of the i^{th} species in the population. PRD and its variants are also extensively studied in economics in game theory as models for dynamical selection of equilibria (see, e.g., [3, 4]). Its generalizations have also been studied in the context of the emergence of organizations in complex adaptive systems (see [5] and references therein). For extensive accounts of more recent contributions to PRD and further references, see the recent books [6, 7].

We view GRD as a model of learning in a community of N interacting agents. The agents are identical in that each is capable of pursuing the same set of strategies with the same payoffs. This is a noncooperative game in which the agents act selfishly (each is concerned with increasing her own payoff without consideration of impact on others or the community) and exhibit bounded rationality (no anticipation of others’ strategy, merely a response to the current aggregate behavior of others). There is no global organizing agency at work; individual actions alone are responsible for the evolution of the system.

Nevertheless, we shall argue that the community as a whole seems to exhibit a kind of global organization under certain circumstances. Individual agents tend to specialize, while the community as a whole retains its diversity (i.e., each pure strategy is pursued by some agent or the other). We attempt to find conditions on the parameters of the model (the size N of the community and the $s \times s$ payoff matrix A) such that this behaviour occurs. While most of the time we work with a strategy space of a fixed size (and refer to diversity as the maintenance of all strategies in this fixed size space), the results also have bearing on the conditions under which new strategies can enter the community.

Section II sets the notation and discusses some relationships between PRD and GRD. Section III identifies conditions under which attractors of GRD can exhibit simultaneously specialization and diversity, and characterize these attractors quantitatively. Section IV summarizes the results, discusses their possible significance, and outlines some open questions. Due to space constraints, proofs for some of the results have not been included in this article.

II. RELATIONSHIPS BETWEEN GRD AND PRD

Notation, Definition of Specialization, and Diversity

Let J denote the simplex of s -dimensional probability vectors:

$$J = \{ \mathbf{x} = (x_1, \dots, x_s)^T \in \mathbf{R}^s \mid \sum_{i=1}^s x_i = 1, x_i \geq 0 \}. \quad (2.1)$$

J is the full configuration space of PRD and is invariant under it.

The configuration space of GRD will be denoted $J^N = \Pi_{\alpha=1}^N J^{(\alpha)}$ where $J^{(\alpha)}$ is a copy of J for the α^{th} agent. A point

of J^N will be denoted $p = (p^1, p^2, \dots, p^s)$, where $p^\alpha = (p_1^\alpha, p_2^\alpha, \dots, p_s^\alpha) \in J^{(\alpha)}$. J^N is invariant under GRD, as the norm of every p^α is preserved under (1.1).

A point of J^N at which every agent has specialized to some strategy or the other will be referred to as a corner of J^N , and at such a point we say the community is “fully specialized.” It is evident that a corner is an equilibrium point of (1.1) since \dot{p}_i^α vanishes if p_i^α does and (1.1) preserves norm; hence, we often refer to a corner as a corner equilibrium point (CEP). A CEP can be characterized by an s -vector of non-negative integers $\mathbf{n} = (n_1, \dots, n_s)$ where n_i is the number of agents pursuing the pure strategy i at the CEP, $1 \leq i \leq s$ (thus $\sum_i n_i = N$). Two CEPs with the same associated \mathbf{n} vector are interchangeable, since they differ only in the identity of the agents, irrelevant for our purposes.

The set $F_k \equiv \{p \in J^N \mid p_k^\alpha = 0 \forall \alpha\}$ for any $k \in S$ is the subset of the boundary of J^N where all agents have opted out of strategy k . At the “face” F_k , strategy k becomes extinct from the population and the full diversity of strategies is lost. The community will be said to exhibit “diversity” at all points that do not belong to some F_k . Note that we use the word “diversity” not to signify the variation between individual agents, but to indicate that all strategies are supported. Indeed we can have no variation but full diversity if all agents pursue the same mixed strategy: For all α , $p^\alpha = \mathbf{c} \in J^\circ$. (The superscript \circ for any set denotes its relative interior.) When p^α is independent of α , the community is completely “homogeneous” since all agents are doing the same thing. The community can be fully specialized and diversified at the same time: Each agent chooses a pure strategy and every strategy is chosen by some agent or the other. This corresponds to CEP with \mathbf{n} such that each n_i is nonzero, which will be called a fully diversified CEP or FDCEP. By contrast, CEP where one or more strategies becomes extinct (some components of \mathbf{n} are zero) will be called non-FDCEP.

In this article, we are primarily interested in studying the circumstances in which FDCEPs are the preferred attractors of the dynamics, since in that case individual specialization and global diversity will arise dynamically in the community.

Differences Between PRD and GRD

If the initial point of a trajectory in GRD is homogeneous, the trajectory remains homogeneous for all time and evolves according to (1.2) except that the time is speeded up by a factor of $N-1$. The sum $\bar{x}_i \equiv (1/N) \sum_{\alpha=1}^s p_i^\alpha$ equals the probability that strategy i is being played in the entire community and is therefore the analogue of x_i in PRD. We can ask how \bar{x}_i evolves in GRD. It is easy to see that

$$\begin{aligned} \dot{\bar{x}}_i &= N[\bar{x}_i \sum_j a_{ij} \bar{x}_j - \sum_{k,j} x_{ik} a_{kj} \bar{x}_j - \\ &\frac{1}{N} \sum_j x_{ij} a_{ij} + \frac{1}{N} \sum_{k,j} x_{ikj} a_{kj}], \end{aligned} \quad (2.2)$$

where $x_{ik} \equiv (1/N) \sum_{\alpha} p_i^\alpha p_k^\alpha$ and $x_{ikj} \equiv (1/N) \sum_{\alpha} p_i^\alpha p_k^\alpha p_j^\alpha$. The r.h.s. of (2.2) is not proportional to the r.h.s. of (1.2), except for homogeneous trajectories in which case $x_{ik} = \bar{x}_i \bar{x}_k$, $x_{ikj} = \bar{x}_i \bar{x}_k \bar{x}_j$. Thus in general \bar{x}_i does not follow the PRD. One might have hoped that when the number of agents N is large \bar{x}_i follows PRD, but even that is not the case due to variation among the agents. For example, at the corner \mathbf{n} , the difference between x_{ik} and $\bar{x}_i \bar{x}_k$ is $n_i(N\delta_{ij} - n_j)/N^2$, which is comparable to the former two even for large N (except for homogeneous corners).

One of our results in this article is that even though variation among agents, which is generic in GRD, causes the evolution of \bar{x}_i to be different from PRD, under suitable conditions \bar{x}_i nevertheless converges to the interior equilibrium point of PRD.

The Interior Equilibria of GRD

Consider an interior equilibrium point (IEP) p of (1.1). By definition, no p_i^α is zero in the interior of J^N . Therefore the bracket $[\]$ on the r.h.s. of (1.1) must vanish for all α, i . Define $x_0^\alpha \equiv \sum_{\beta \neq \alpha} \sum_{i,j} p_i^\alpha a_{ij} p_j^\beta$, and $v_i^\alpha \equiv \sum_{\beta \neq \alpha} p_i^\beta$. Then $\sum_{i=1}^s v_i^\alpha = N-1 \forall \alpha$, and the interior equilibrium condition can be written as

$$BX^\alpha = (N-1)E_0 \forall \alpha, \quad (2.3)$$

where $X^\alpha \equiv (x_0^\alpha, v_1^\alpha, v_2^\alpha, \dots, v_s^\alpha)^T$, B is the $s+1$ -dimensional matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & & & & \\ -1 & A & & & \\ \vdots & & & & \\ -1 & & & & \end{pmatrix}, \quad (2.4)$$

and E_0 is the $s+1$ -dimensional unit vector $(1, 0, 0, \dots, 0)^T$.

It is not difficult to see that (2.3) has an isolated solution if and only if the following condition holds:

A1: $u_i \neq 0 \forall i$, and all u_i have the same sign, where u_i denotes the co-factor of B_{0i} .

For the sake of notational simplicity, we have denoted the co-factor of B_{0i} by u_i instead of u_{0i} , thereby suppressing the fixed first index. Under the above condition (A1) $\det B = \sum_{i=1}^s u_i \neq 0$, and the solution is unique and given by $v_i^\alpha = (N-1)x_i \forall \alpha$. Here,

$$x_i = u_i / \det B \quad (2.5)$$

is nothing but the i^{th} coordinate of the unique isolated interior equilibrium point of PRD. (Note that A1 is also the necessary and sufficient condition for PRD to have an isolated IEP,

which, if it exists, is unique.) Since $p_i^\alpha - p_i^\beta = v_i^\beta - v_i^\alpha = 0$, it follows that the equilibrium point is homogeneous and given by $p_i^\alpha = x_i$. Thus we have proved

Theorem 2.1 There exists at most one isolated equilibrium in the interior of J^N . It exists if and only if A1 is satisfied and then it is homogeneous (all agents pursue the same mixed strategy), and coincides with the isolated interior equilibrium point of PRD, $p_i^\alpha = x_i \forall \alpha, i$.

III. DIVERSIFICATION WITH SPECIALIZATION

Stability of corner equilibria

The IEP of GRD is always unstable to small perturbations. This is a consequence of the following theorem proved in [1]: **Theorem 3.1** An equilibrium point of (1.1) is asymptotically stable if and only if it is a strict Nash equilibrium. Further, any compact set in the relative interior of a face cannot be asymptotically stable.

Note that strict Nash equilibria are perforce pure strategy Nash equilibria and therefore correspond to CEP. As a consequence of this theorem, a trajectory either eternally moves around in the relative interior of some face or the interior of J^N coming arbitrarily close to its boundaries and corners (the case of noncompact attractor), or it converges to a corner of J^N . It is possible to construct payoff matrices for which there are no asymptotically stable corners in J^N , whereupon the former situation obtains.

However, our numerical work with 3×3 payoff matrices suggests that this happens rarely (i.e., in a relatively small region of $\mathbf{R}^{3 \times 3}$) for most payoff matrices, asymptotically stable corners do exist for most values of N . Further, we randomly generated ten 3×3 payoff matrices and numerically integrated the GRD equations for long times for each payoff matrix with ten randomly chosen initial conditions. When this was done with $N = 5$, in 90 out of the 100 cases the dynamics converged to a corner. With $N = 10$, all 100 cases converged to a corner.

This suggests that typically, at large N , not only do asymptotically stable corners exist, but also their basins of attraction cover most of J^N . Thus, corners seem to be the most common attractors in GRD. These are numerical indications and need to be made more precise. In our interpretation of the model, a corner corresponds to a fully specialized community. The above theorem and numerical evidence, therefore, suggest that specialization of all the agents is the most common outcome in GRD.

At the CEP \mathbf{n} , the payoff to an agent playing the j^{th} pure strategy from the other $N-1$ agents is

$$P_j = \sum_{k \neq j}^s a_{jk} n_k + (n_j - 1) a_{jj} = \sum_{k=1}^s a_{jk} n_k - a_{jj} = P_j - a_{jj}, \quad (3.1)$$

where $P_j \equiv \sum_{k=1}^s a_{jk} n_k$. If this agent were to suddenly switch to the i^{th} pure strategy ($i \neq j$), all other agents remaining at their respective pure strategies, then for this agent the payoff would

change to $\sum_{k \neq j}^s a_{ik} n_k - a_{ij} (n_j - 1) = P_i - a_{ij}$. Thus, the increase in payoff for an agent playing the j^{th} pure strategy at the FDCEP \mathbf{n} (and this assumes $n_j \neq 0$) in switching to the i^{th} pure strategy is

$$\lambda_{ij} = P_i - P_j - h_{ij}, \quad h_{ij} \equiv a_{ij} - a_{jj}. \quad (3.2)$$

Therefore, \mathbf{n} is a strict Nash equilibrium if for every j such that $n_j \neq 0$, the conditions

$$\lambda_{ij} < 0 \quad (3.3)$$

are satisfied for all $i \neq j$. At a FDCEP, all n_j are nonzero and this is a set of $s(s-1)$ conditions. At a non-FDCEP where only $s' < s$ components of \mathbf{n} are nonzero, the number of conditions is smaller, $s'(s-1)$.

From Theorem 3.1, these are identical to the conditions for the asymptotic stability of the FDCEPs associated with \mathbf{n} . In fact, one can show that λ_{ij} given by (3.2) are precisely the eigenvalues of the Jacobian matrix of (1.1) linearized around a corner of J^N characterized by \mathbf{n} .

Stability of fully diversified corners

Theorem 3.2: Let PRD admit an isolated IEP \mathbf{x} . That is, condition A1 holds [cf. Section II]. Let \mathbf{n}, \mathbf{n}' be any pair of asymptotically stable FDCEPs of GRD with $N \geq s$. Then (1) all components of the difference $\mathbf{n}' - \mathbf{n}$ are bounded by a function of A alone, not of N , and (2) $\lim_{N \rightarrow \infty} \frac{n_i}{N} = x_i$. The proof is given in Appendix A.

The significance of this theorem is that it characterizes the FDCEP that are attractors of the dynamics. If the community is going to end up in a fully specialized and diversified configuration, the theorem quantifies the relative weights of all strategies that will obtain in that configuration: These relative weights are forced to be "close" to the IEP configuration given by (2.5). The theorem does not guarantee the existence of a stable FDCEP. One can prove the existence of an infinite set of values of N at which stable FDCEP are guaranteed to exist under the conditions of the theorem. One can also identify sufficient conditions for the existence of stable FDCEP for any $N \geq s$.

Instability of non-fully diversified corners

We would like to define GRD as possessing diversity if all trajectories in the faces F_k become unstable at some time or the other with respect to perturbations that take them away from these faces. With this in mind, we now study corners at which one or more strategies become extinct and determine the conditions under which all such corners become unstable. Then under small perturbations, the population will dynamically flow out of such corners, eliminating specialized configurations that do not carry the full diversity of strategies.

As mentioned earlier, the number of conditions to be satisfied by a non-FDCEP to be stable is less than the number to

be satisfied by a FDCEP. Thus a priori, things seem to be loaded against diversification. As we shall see, some further structure will need to be imposed on A in order to make the non-FDCEP unstable. At this point, we do not have the general conditions for arbitrary s , but some insight gleaned from special cases $s = 2, 3$.

$s = 2, N$ arbitrary

In this case conditions (3.3) can be studied exhaustively. There are generically four cases:

Case 1: $a_{11} > a_{21}$ and $a_{22} > a_{12}$: Both $(N, 0)$ and $(0, N)$ are asymptotically stable, other corners are not.

Case 2: $a_{11} > a_{21}$ and $a_{22} < a_{12}$ ($N, 0$) is the only asymptotically stable corner.

Case 3: $a_{11} < a_{21}$ and $a_{22} > a_{12}$: ($0, N$) is the only asymptotically stable corner.

Case 4: $a_{11} < a_{21}$ and $a_{22} < a_{12}$: The only asymptotically stable corners (n_1, n_2) (with $n_1 + n_2 = N$) are those for which $n_1 \neq 0$, $n_2 \neq 0$, and furthermore,

$$\frac{(n_2 - 1)}{n_1} h_{12} < h_{21} < \frac{n_2}{(n_1 - 1)} h_{12} \text{ if } n_2 < N - 1, \text{ and } \frac{(n_2 - 1)}{n_1} h_{12} < h_{21} \text{ if } n_2 = N - 1.$$

Cases 2 and 3 correspond to dominated strategies. (The cases with one or more equalities instead of inequalities have been disregarded as nongeneric. In any case, they are not difficult to handle.) The case of interest to us is the last one, which shows diversification. It is convenient to introduce the

Definition: A is diagonally subdominant if $a_{ii} < a_{ji} \forall j \neq i, \{i, j\} \subset S$.

That is, $h_{ij} > 0 \forall i \neq j$. From the above exhaustive list it follows that the condition

A2: A is diagonally subdominant

is the necessary and sufficient condition for non-FDCEP to be unstable (for generic A). If **A2** is satisfied, the only asymptotically stable CEP are the FDCEP, for which Theorem 3.2 applies. (Note that for $s = 2$, **A2** implies **A1**, the IEP is given by

$$p^\alpha = \frac{1}{h_{12} + h_{21}} (h_{12}, h_{21})$$

for all α , and the inequalities involving \mathbf{n} in Case 4 are equivalent to the statement that $(1/N)(n_1, n_2)$ must be close to this IEP for arbitrary N and converge to it as $N \rightarrow \infty$.)

Note that GRD remains invariant under addition of an arbitrary constant to any column of the payoff matrix. Thus, we may replace a_{ij} by h_{ij} , obtaining a matrix which under **A2** has zero diagonal elements and nonnegative off-diagonal elements. It is interesting that these conditions also arise in PRD

in the context of population genetics and ecological models [8] as well as in models of catalytic networks of chemically reacting molecules [9].

$s = 3, N$ arbitrary

For $s = 3$, **A2** no longer implies **A1**; the latter is an independent condition. We now state **Theorem 3.3:** For $s = 3$, if both **A1** and **A2** hold, then there exists a positive number N_0 depending on A such that for all $N > N_0$, all non-FDCEP are unstable. The proof of this theorem can be found in Appendix B.

We remark that while N_0 is finite, it may, depending on A , be much larger than three. The above result can be further generalized (with the imposition of an additional condition) to prove that for $s = 3$ all points in F_k are unstable for sufficiently large N . Note that our notion of diversity for GRD is related to the notions of "permanence," "persistence," etc. introduced for PRD (see [2] and references therein).

PRD is said to exhibit permanence if every interior solution has components that remain bounded away from zero by a common constant $\delta > 0$. Strong persistence, in turn, is the weaker requirement where δ is trajectory dependent and persistence the even weaker requirement that each component of an interior trajectory not converge to zero. The biological implications are obvious: The concept is clearly related to survival of species. The corresponding phenomenon here is the survival of policies. The conditions, for example, for permanence in PRD (see [2]) may quite generally play a role in discussions of diversity in GRD.

IV. DISCUSSION AND CONCLUSIONS

We have considered the equation (1.1) as a model of evolution of a community of N agents, each agent being capable of performing any mix of a set of s strategies and modifying its mix depending on the payoff received from other agents. We have studied some properties of the attractors of this system to gain insight into how the community is expected to evolve.

This model can exhibit specialization of the agents into pure strategies. Evidence for this comes from the previously known Theorem 3.1, supported with our numerical observations. While individual specialization seems to be the most common outcome in this model, it would be interesting to characterize more precisely the circumstances in which specialization is guaranteed (i.e., when corners are the only attractors, and when they are not.)

We have shown that under suitable conditions, while each agent specializes to a single pure strategy, it is guaranteed that the community as a whole preserves the full diversity of strategies. These are that the community be sufficiently large (N should be larger than a number N_0 that depends on the payoff matrix), and the payoff matrix itself should satisfy **A1** (existence of an isolated interior equilibrium point) and **A2** (diagonal entries of A be smaller than other entries in the same column). These guarantee (for up to three strategies) that all corners

where one or more strategy becomes extinct are unstable to small perturbations (Theorem 3.3). To identify sufficient conditions for larger s (and necessary and sufficient conditions for $s \geq 3$) is a task for the future. The appearance of a lower limit on the size of the community in this context (which could be much larger than the number of strategies) is interesting.

Within the set of configurations where the community would exhibit full specialization and diversity (the FDCEP), we have given a quantitative criterion as to which ones will be the attractors (Theorem 3.2). n_i/N (where n_i is the number of agents pursuing the pure strategy i at the attractor) is forced to be close to x_i and equal to it in the large N limit, where x_i is given by (2.5) and is the relative weight of the i^{th} strategy at the interior equilibrium point of PRD. This constraint is a consequence of the fine balance that exists for every agent at a strict Nash equilibrium; any strategy switch for any agent reduces its payoff. This fine-tuning, caused by the interaction of the agent with other agents, is a kind of organization exhibited by the system.

The conditions for the instability of non-FDCEP (Theorem 3.3) may also be relevant to the question: When does a society accept an innovation? Consider a community of a large number of agents but with only two strategies, 1 and 2, at a stable corner where n_1 agents pursue the pure strategy 1 and $n_2 = N - n_1$ agents the pure strategy 2 (neither n_1 nor n_2 is zero). Since this corner is assumed stable, the 2×2 matrix A satisfies condition **A2** (diagonal subdominance). Now imagine that a new strategy 3 arises, thereby enlarging the payoff matrix to a 3×3 matrix A' containing A as a 2×2 block. In the new context the earlier state of the community will be described by a three vector $\mathbf{n} = (n_1, n_2, 0)$, which is in the face F_3 . Now if the new payoff matrix satisfies **A1**, **A2**, and N is sufficiently large, then from Theorem 3.3, this configuration is unstable with respect to perturbations in which one of the agents begins to explore the new strategy. Thus, if this agent were to explore the new strategy ever so slightly, the payoff would increase and a small perturbation of the community would grow until it settles down in another attractor. The new attractor, if described by Theorem 3.2, would have the property that a finite fraction of the population pursues the new strategy: The innovation has been accepted by the society. Thus, the conditions **A1**, **A2** of Theorem 3.3 indicate what the payoffs of a new strategy (innovation) should be with respect to the existing ones, if the innovation is to be guaranteed acceptance. (Conditions that are both necessary and sufficient for diversity would considerably strengthen the above remarks.)

It is worth mentioning that conditions **A1**, **A2** are not equalities but inequalities. Thus, there is no fine-tuning of parameters needed; the behavior discussed above emerges whenever parameters cross certain thresholds.

It may be interesting to consider the “economic significance” of conditions that play an important role in preserving the full diversity of strategies. For example, diagonal subdominance, when translated as “each pure strategy gives more payoff to

other pure strategies than to itself,” carries a shade of an altruism of sorts (at the level of strategies, not individuals). Note that PRD with a payoff matrix in which diagonal entries are zero and offdiagonal ones greater than or equal to zero is called a “catalytic network” [2]. The general message might be that if the initial set of allowed strategies is chosen with the “right vision” (read “right payoffs”), then even a community of identical and selfish individuals, if large enough, will exhibit diversity and accept only the “right” innovations.

APPENDIX A

The proof of Theorem 3.2 follows:

Proof: Note that $P_i - P_j$ figures in both λ_{ij} and λ_{ji} . Therefore, the $s(s-1)$ conditions (3.3) can be written in terms of $s(s-1)/2$ “double-sided” inequalities

$$-h_{ji} < P_i - P_j < h_{ij}. \quad (4.1)$$

Define $z_i \equiv P_i - P_{i+1}$ for $i = 1, \dots, s$, with $P_{s+1} \equiv P_1$. Then $z_i = \sum_{j=1}^s c_{ij} n_j$ with $c_{ij} \equiv a_{ij} - a_{i+1,j}$, where it is again understood that $a_{s+1,j} \equiv a_{1j}$. Now, since all the n_i are not independent, let us express z_i in terms of only n_1, \dots, n_{s-1} by eliminating $n_s = N - (n_1 + \dots + n_{s-1})$. This gives $z_i = y_i + c_{is}N$ where $y_i \equiv \sum_{j=1}^{s-1} d_{ij} n_j$ and $d_{ij} \equiv c_{ij} - c_{is}$, $i, j = 1, \dots, s-1$. With this notation, consider the subset of $s-1$ inequalities obtained by setting $j = i+1$ in (4.1), with $i = 1, \dots, s-1$. These involve z_i and take the form

$$-h_{i,i+1} - c_{is}N < y_i < h_{i,i+1} - c_{is}N, \quad i = 1, \dots, s-1. \quad (4.2)$$

These inequalities mean that for any stable FDCEP \mathbf{n} , the y_i , which are linear combinations of n_1, \dots, n_{s-1} , are constrained to be in an open interval of the real line. While the location of this interval is N dependent, it follows from (4.2) that the size of this interval is finite, independent of N , and depends only on the payoff matrix (for y_i , the size of the interval is $h_{i,i+1} + h_{i,i+1}$).

If the $s-1$ dimensional matrix $D = (d_{ij})$ has an inverse, we can invert $y_i \equiv \sum_{j=1}^{s-1} d_{ij} n_j$ to express the n_j in terms of y_i . Then, (4.2) will get converted into inequalities for n_1, \dots, n_{s-1} . Since the vector $\tilde{\mathbf{y}} = (y_1, \dots, y_{s-1})$ in the $s-1$ dimensional cartesian space whose axes are the y_i is constrained by (4.2) to lie in a (rectangular) parallelepiped, the vector $\tilde{\mathbf{n}} = (n_1, \dots, n_{s-1})$ in the $s-1$ dimensional cartesian space whose axes are the n_i will also lie in a (in general oblique) parallelepiped which is the image, under D^{-1} , of the rectangular parallelepiped in y -space defined by (4.2). Again, while the location of the parallelepiped in n_1, \dots, n_{s-1} space will depend on N , its size (i.e., its extent along any of the coordinate axes) will be independent of N . This is because the matrix D^{-1} , if it exists, depends only on the payoff matrix and not on N . Therefore, if D^{-1} exists, the differences in n_i , $i = 1, \dots, s-1$ for all FDCEP are bounded by

some function of A alone, not of N . The same is true for n_s , also since $\sum_{i=1}^s n_i = N$. One can show that $\det D = \det B$. The existence of D^{-1} is thus guaranteed by condition **A1**, completing the proof of the first part of Theorem 3.2.

To prove the second part of Theorem 3.2, divide all sides of (4.1) by N and take the limit $N \rightarrow \infty$. This yields

$$\lim_{N \rightarrow \infty} \left[\frac{P_i}{N} - \frac{P_j}{N} \right] = 0.$$

Defining $x'_i \equiv \lim_{N \rightarrow \infty} (n_i/N)$ along an appropriate subsequence independent of i , this is equivalent to the statement that $\sum_{k=1}^s a_{ik} x'_k$ is independent of i , which implies that x' is the same as x , the IEP of PRD.

APPENDIX B

The proof of Theorem 3.3 follows:

Proof: Any non-FDCEP \mathbf{n} must belong to some F_k , in this case to F_1, F_2 or F_3 . For every i such that the component n_i of $\mathbf{n} \in F_k$ is nonzero, consider the eigenvalue

$$\lambda_{ki} = P_k - P_i - h_{ki}. \quad (4.3)$$

From the discussion of Eq. (3.2), it follows that if any one (or the largest) of the λ_{ki} at \mathbf{n} is greater than zero, then the CEP \mathbf{n} is unstable against perturbations in which an agent pursuing the pure strategy i moves toward strategy k (i.e., the perturbations which restore the extinct strategy k will then grow).

For concreteness consider F_3 . Corners of F_3 are of two types:

Case 1: Only one strategy survives at the corner. Then $\mathbf{n} = (N, 0, 0)$ or $(0, N, 0)$. In the former case (4.3) implies $\lambda_{31} = (N-1)h_{31}$ and in the latter case $\lambda_{32} = (N-1)h_{32}$. By **A2** both corners are unstable.

Case 2: Both strategies 1 and 2 survive at the corner of F_3 . Then $\mathbf{n} = (n_1, n_2, 0)$ with both n_1 and n_2 positive integers and $n_1 + n_2 = N$. There are then two eigenvalues from (4.3), $\lambda_{31} = h_{31}n_1 + h_{32}n_2 - h_{12}n_2 - h_{31}$, and $\lambda_{32} = h_{31}n_1 + h_{32}n_2 - h_{21}n_1 - h_{32}$. Let us assume that this corner is stable; hence, both $\lambda_{31}, \lambda_{32}$ are negative. The condition $\lambda_{31} < 0$ (upon eliminating $n_1 = N - n_2$) reduces to $(h_{12} + h_{31} - h_{32})n_2 > (N-1)h_{31}$. Since $n_2, N-1$, and h_{31} are all positive, this means the combination $h_{12} + h_{31} - h_{32}$ is also positive, and

$$\frac{(N-1)h_{31}}{h_{12} + h_{31} - h_{32}} < n_2. \quad (4.4)$$

Similarly $\lambda_{32} < 0$ implies that $h_{21} + h_{32} - h_{31}$ is positive (as can be seen by eliminating n_2) and further,

$$n_2 < \frac{(N-1)(h_{21} - h_{31})}{h_{21} + h_{32} - h_{31}} + 1. \quad (4.5)$$

Combining the two, we get

$$\frac{(N-1)h_{31}}{h_{12} + h_{31} - h_{32}} < \frac{(N-1)(h_{21} - h_{31})}{h_{21} + h_{32} - h_{31}} + 1, \quad (4.6)$$

which can be rearranged into the form

$$(N-1)[-h_{21}h_{12} + h_{21}h_{32} + h_{31}h_{12}] < (h_{12} + h_{31} - h_{32})(h_{21} + h_{32} - h_{31}). \quad (4.7)$$

But the quantity in [] on the l.h.s. of this inequality is just u_3 (as evaluated from the definition given in **A1**), which is positive. (The positivity of $\det B$ and hence u_1, u_2, u_3 also follows from **A1** and **A2**.) Thus, we have

$$N < \frac{(h_{12} + h_{31} - h_{32})(h_{21} + h_{32} - h_{31})}{u_3} + 1. \quad (4.8)$$

Note that the r.h.s. is a function of A alone and is finite, say $N_0(A)$. If N is chosen larger than $N_0(A)$, this inequality is violated. That is, for $N > N_0(A)$, the corner of F_3 under consideration cannot be stable. We have thus proved that under **A1, A2**, all corners of F_3 are unstable for $N > N_0(A)$. Similarly one may consider F_1, F_2 , which will yield the same result but with different finite bounds in place of $N_0(A)$. We can henceforth use N_0 for the largest of the three. The claim follows.

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