Applied Mathematics and Computation 217 (2010) 2495–2500

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Invariant norm quantifying nonlinear content of Hamiltonian systems

Govindan Rangarajan [a,1], Shrihari Sridharan [b,*]

[a] Indian Institute of Science, Bangalore 560 012, India
[b] Chennai Mathematical Institute, Chennai 603 103, India

ARTICLE INFO

Keywords:
Hamiltonian systems
Symplectic maps
Invariant norm

ABSTRACT

Given a Hamiltonian system, one can represent it using a symplectic map. This symplectic map is specified by a set of homogeneous polynomials which are uniquely determined by the Hamiltonian. In this paper, we construct an invariant norm in the space of homogeneous polynomials of a given degree. This norm is a function of parameters characterizing the original Hamiltonian system. Such a norm has several potential applications.

1. Introduction

Hamiltonian systems form an important class of dynamical systems. It is important to be able to quantify the nonlinear content of Hamiltonian systems through an appropriate quantity. This can then serve as a merit function for optimizing the performance of the Hamiltonian system as a function of the parameters characterizing the Hamiltonian system. In particular, one can try to maximize the stability region of the system (in the context of particle accelerators [6]), minimize optical aberrations (in the context of optical systems [8]), etc. We quantify the nonlinear content of the Hamiltonian system using its corresponding symplectic map.

Symplectic maps and invariants have been successfully used to better understand the Hamiltonian systems [1–3,5,6, 9–11,13–20,26,27]. In fact, the time evolution of the Hamiltonian system with $2n$ degrees of freedom can be directly described by the symplectic map, say $\mathscr{M}$, generally specified by a set of homogenous polynomials in the $2n$ phase space variables of the Hamiltonian system and of a certain degree [9]. The nonlinear content of $\mathscr{M}$ (and hence of the original Hamiltonian system) is given by homogeneous polynomials of degree greater than two. However, this representation is not very useful to explicitly compute the nonlinear content as it involves an infinite series. This limitation is overcome by employing Lie perturbation theory where we consider the terms degree by degree in the space of homogenous polynomials. Further, we require that the quantity characterizing the nonlinear content be invariant under the action of an appropriate symmetry group. Here, the underlying symmetry group for the full $\mathscr{M}$ turns out to be an infinite dimensional, non-compact, Lie group. This implies that it is not possible to construct a quantity invariant under the full symmetry group. Consequently, rather than quantifying and minimizing the entire nonlinear content in $\mathscr{M}$, we will in fact restrict ourselves to doing the same for symplectic maps with homogeneous polynomials truncated at degree 3. This is in line with the perturbative approach that we have adopted since the leading order nonlinearity in $\mathscr{M}$ comes from homogenous polynomials of degree 3. Furthermore, as we shall see later, for a quantity to remain invariant under the action of the above truncated symplectic map, it is sufficient to demand that it remains invariant under the linear part of the symplectic map since the nonlinear term contributes only a fourth order correction that we can ignore (in the spirit of perturbation theory) having truncated at degree 3. By this process, we have also reduced the dimension of the underlying symmetry group to be the finite dimensional real

* Corresponding author.
E-mail addresses: rangaraj@math.iisc.ernet.in (G. Rangarajan), shrihari@cmi.ac.in (S. Sridharan).
[1] This author was supported by DST Centre for Mathematical Biology (SR/S4/ MS:419/07) and UGC-SAP (Phase IV).

0096-3003/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2010.07.060

symplectic group $Sp(2n,\mathbb{R})$. Since even this group is non-compact, we still are not in a position to define any quantity that is invariant under its action. To get around this problem, we put the truncated symplectic map into what is known as its "normal form" using a symplectic (canonical) transformation. Now, the symmetry group for the normal form is the group of real symplectic orthogonal matrices. We note that this is isomorphic to the compact unitary group $U(n)$. Hence, we can now restrict ourselves to finding a quantity in the space of homogenous polynomials of degree 3 in the $2n$ phase space variables that is invariant under the action of the above unitary group. This is accomplished by performing an invariant integration over $U(n)$ of a suitable function. An earlier, approximate treatment can be found in [28]. Once we have an invariant, we can use this as a merit function to determine optimal values of the system parameters and consequently, enhance the "performance" of the system.

2. Symplectic maps and homogeneous polynomials

Through our discussion in the first section, it is quite clear that the homogenous polynomials of the phase space variables play a vital role in the Lie perturbation theory of Hamiltonian dynamics. In this section, we shall fix notations and build the theory to that extent in order that we give a meaningful representation of any homogenous polynomial of a certain degree in terms of the basis monomials of the appropriate space.

Consider the Hamiltonian system given by $2n$ phase space variables that we denote by

$$z = (q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n).$$

For any fixed phase space function $f(z)$, let $\mathscr{L}_f$ denote the corresponding Lie operator defined on the space of phase space functions by

$$\mathscr{L}_f g := [f,g] = fg - gf.$$

It is then a simple observation that the above defined bracket is anti-symmetric and linear that satisfies $[f,(gh)] = [f,g]h + g[f,h]$. The exponential of the above defined operator is called the Lie transformation that is again defined on the space of phase space functions.

$$(e^{\mathscr{L}_f})(g) := \left(\sum_{n\geqslant 0}\frac{\mathscr{L}_f^n}{n!}\right)(g) = \frac{g}{0!} + \frac{[f,g]}{1!} + \frac{[f,[f,g]]}{2!} + \frac{[f,[f,[f,g]]]}{3!} + \cdots.$$

The effect of the Hamiltonian system on a particle can be formally expressed as the action of a map $\mathscr{M}$ that takes the particle from its initial state $z_{\rm in}$ to its final state $z_{\rm fin}$, i.e., $z_{\rm fin} = \mathscr{M} z_{\rm in}$. It can be shown that $\mathscr{M}$ is a symplectic map, [6,9]. Symplectic maps are maps whose $2n \times 2n$ Jacobian matrices $M(z)$ satisfy the 'symplectic condition' given by $\widetilde{M(z)}JM(z) = J$, where $\widetilde{M}$ represents the transpose of M and J is the fundamental symplectic matrix given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Here I is an $n \times n$ identity matrix. The set of all symplectic matrices form the real symplectic group $Sp(2n,\mathbb{R})$. This finite dimensional non-compact real symplectic Lie group is the underlying symmetry group for the linear part of the symplectic map $\mathscr{M}$ with n degrees of freedom. The symplectic map can be factorised using the result due to Dragt and Finn.

Lemma 1 (Dragt–Finn Factorisation Theorem, [7]). *The symplectic map $\mathscr{M}$ can be factorised as*

$$\mathscr{M} = \widehat{M} e^{\mathscr{L}_{f_3}} e^{\mathscr{L}_{f_4}} \cdots e^{\mathscr{L}_{f_n}} \cdots,$$

where, $\widehat{M}$ gives the linear part of the map while the infinite product of Lie transformations $e^{\mathscr{L}_{f_n}}, n = 3,4,\ldots$ represents the nonlinear part of $\mathscr{M}$. Here, f_n denotes a unique homogenous polynomial in the phase space variables of degree n.

Observe that the linear part $\widehat{M}$ of the symplectic map $\mathscr{M}$ has an equivalent representation in terms of the Jacobian matrix M of the map $\mathscr{M}$,

$$\widehat{M} z_i = M_{ij} z_j = (Mz)_i.$$

We now undertake the task of indexing the basis monomials appropriately. Although this is a simple task, we urge the reader to look at its significance in the later sections. Let us denote by $\mathscr{P}^{(m)}$, the space of all homogenous polynomials in z of degree m. Let $\left\{P_\alpha^{(m)}\right\}$ be the basis for this space. By a result due to Nijenhuis and Wilf [24], we know that the dimension of this space is given by

$$N(2n,m) = \binom{2n+m-1}{m}.$$

We take the basis $P_\alpha^{(m)}(z)$ to be the basis monomials of degree m in the $2n$ variables,

$$P_\alpha^{(m)}(z) = q_1^{r_1} p_1^{r_2} \cdots q_n^{r_{2n-1}} p_n^{r_{2n}}; \quad 1 \leqslant r_i \leqslant m, \ \Sigma_i r_i = m.$$

Adopting the method suggested by Giorgilli in [12], we now associate to each basis monomial, a convenient numerical index. Let

$$i(r_1, r_2, \ldots, r_{2n}) = \sum_{j=1}^{2n} \begin{pmatrix} j - 1 + \sum\limits_{k=0}^{j-1} r_{2n-k} \\ j \end{pmatrix}.$$

One can then easily observe that q_1^m is the first monomial with degree m while p_n^m is the last monomial, i.e.,

$$\begin{aligned} i(m, 0, \ldots, 0, 0) &= \min i(r_1, r_2, \ldots, r_{2n}); \\ i(0, 0, \ldots, 0, m) &= \max i(r_1, r_2, \ldots, r_{2n}); \quad \Sigma\, r_i = m. \end{aligned} \tag{1}$$

Also observe that Eq. (1) gives the total number of monomials of degree m. Hence, the following definition of α makes sense.

Lemma 2. *Define a value α for the elements in the basis monomials $\left\{P_\alpha^{(m)}\right\}$ by*

$$\alpha(r_1, r_2, \ldots, r_{2n}) = i(r_1, r_2, \ldots, r_{2n}) - i(0, 0, \ldots, 0, m-1).$$

Then α defines a unique number between 1 and $N(2n, m)$ for each of the elements in $\left\{P_\alpha^{(m)}\right\}$.

The above defined quantity α could well provide us an index for the basis monomials. Thus, any polynomial $f_m(z)$ is spanned by a linear combination of elements in the basis and hence has the representation;

$$f_m(z) = \sum_{\alpha=1}^{N(2n,m)} c_\alpha P_\alpha^{(m)}(z),$$

where the quantities c_α are real constants.

We shall confine our interest, for purposes of this paper to single particle dynamics (with 3 degrees of freedom). The construction of an arbitrary n-dimensional unitary transformation $U(n)$ has been discussed in sufficient detail in Poźniak et al. [25] among other papers. Our aim is to go through $U(3)$ (the relevant group) to complete our calculations. We shall remark that the product of the volume element by any positive constant still remains the element of volume of that appropriate unitary group. In fact, we shall normalize the element of volume of the n-dimensional unitary group by multiplying to it a factor of $(1/2\pi)^n$.

3. Quantification of the nonlinear content

In this section, we define a norm that is invariant under an appropriate symmetry group and moreover quantifies the nonlinear content of the Hamiltonian system. We start with the nonlinear Hamiltonian. We assume that it depends on certain parameters collectively denoted by s. Our goal is to obtain optimal values for these parameters so that the "performance" of the system is enhanced. For example, we may wish to maximize the stability region around the operating point of the system in phase space by optimizing the above parameter values.

We represent the action of the Hamiltonian system on a particle using the symplectic map $\mathscr{M}$. Next, we factorize $\mathscr{M}$ using the Dragt–Finn factorization as

$$\mathscr{M} = \widehat{M} e^{\mathscr{L}_{f_3}} e^{\mathscr{L}_{f_4}} \cdots e^{\mathscr{L}_{f_n}} \cdots.$$

Explicit expressions for $\widehat{M}$ and f_n can be obtained from the Hamiltonian using a standard procedure [7,9]. Consequently, even $\widehat{M}$ and f_n now depend on s (we have suppressed this dependence for notational convenience). Since the above factorization cannot be explicitly evaluated, we first truncate the symplectic map to degree 3 (thus retaining only the leading order of nonlinearity):

$$\mathscr{M} \approx \mathscr{M}_3 = \widehat{M} e^{\mathscr{L}_{f_3}}.$$

Thus we are restricting ourselves to quantifying only the leading order of nonlinearity. This suffices for most practical problems since it is this leading order which contributes maximally to the nonlinearity.

We would like the quantity characterizing the nonlinear content to be invariant under the action of $\mathscr{M}_3$. Now the symmetry group for the linear part $\widehat{M}$ for n degrees of freedom is the finite dimensional non-compact Lie group $Sp(2n, \mathbb{R})$. A quantity that is invariant under the linear part would also be invariant under $\mathscr{M}_3 = \widehat{M} e^{\mathscr{L}_{f_3}}$, since the nonlinear term contributes only a fourth order correction. Therefore, the relevant symmetry group for our purpose would be $Sp(2n, \mathbb{R})$. However, as $Sp(2n, \mathbb{R})$ is non-compact, there can be no metric that is invariant under the action of this group. To get around this problem, we first convert $\widehat{M}$ into its so-called "normal form" using a symplectic transformation.

Let A be the symplectic transformation that takes M into its normal form $N = A^{-1}MA$. The normal form N is a block diagonal matrix with each block a rotation in position-momentum plane. This is equivalent to transforming to action-angle variables in conventional canonical perturbation theory. See [9] for further details. The normal form $\mathscr{N}_3$ of the symplectic map $\mathscr{M}_3$ is given by

$$\mathcal{N}_3 = A^{-1}\mathcal{M}_3 A = A^{-1}MAA^{-1}e^{\mathcal{L}_{f_3}}A = Ne^{\mathcal{L}_{f_3^{tr}}},$$

where

$$f_3^{tr} = A^{-1}f_3(z) = f_3(A^{-1}z).$$

Since A is dependent upon M, so is f_3. The homogenous polynomial f_3^{tr} can be expressed as a linear combination of the basis monomials as

$$f_3^{tr}(z) = b_\alpha P_\alpha^{(3)}(z),$$

where b_α are real constants. Note that b_α would depend on the parameters s. The symmetry group for N (and consequently for $\mathcal{N}_3$) is the group of real symplectic orthogonal matrices and this is isomorphic to the unitary group $U(n)$ [9].

We now provide an explicit construction of a quantity that characterizes the leading order nonlinearity of the symplectic map (and hence the Hamiltonian system that this map represents). First, we construct an invariant metric on the space of homogenous polynomials of a given degree m in $2n$ phase space variables. By its very name, it is clear that we are looking for a symmetric, positive definite bilinear form that is invariant under the action of $U(n)$. Let $\mathcal{G}$ denote either $U(2)$ or $U(3)$. We define the bilinear form by

$$g^{(m)}_{\alpha_1,\alpha_2} \equiv \left(P^{(m)}_{\alpha_1}(z), P^{(m)}_{\alpha_2}(z)\right).$$

We require $g^{(m)}_{\alpha_1,\alpha_2}$ to be invariant under the action of $\mathcal{G}$, i.e.,

$$\left(UP^{(m)}_{\alpha_1}(z), UP^{(m)}_{\alpha_2}(z)\right) = \left(P^{(m)}_{\alpha_1}(z), P^{(m)}_{\alpha_2}(z)\right),$$

where $U \in \mathcal{G}$. Here, U has to be embedded in $Sp(4,\mathbb{R})$ or $Sp(6,\mathbb{R})$, as the case may be before it can act on $P^{(m)}_\alpha(z)$ [26].

The standard way of constructing an invariant metric is to use the invariant integral from group theory, as explained by Cornwell in [4]. For a Lie group, it is defined as

$$I = \int_{\mathcal{G}} h(U)\sigma(U)dU,$$

where $U \in \mathcal{G}, h(U)$ is a function defined on $\mathcal{G}$ and $\sigma(U)$ is the Haar measure for the Lie group $\mathcal{G}$. In our case of study, the matrix U is taken as mentioned in the last section. Since we are interested in symmetric positive definite invariant bilinear forms on the space of homogenous polynomials of degree m, it is natural to take

$$h(U) = [\mathcal{D}^m]^T(U)\mathcal{D}^m(U),$$

where $\mathcal{D}^m$ is a square matrix of order $N(2n,m)$ defined by

$$UP^m_\alpha(z) = P^m_\alpha(Uz) = \sum_{i=1}^{N(2n,m)} \mathcal{D}^m(U)^i_\alpha P^m_i(z),$$

and $[\mathcal{D}^m]^T$ is the transpose of $\mathcal{D}^m$. An elementary result in matrix theory shows that $[\mathcal{D}^m]^T(U)\mathcal{D}^m(U)$ is both symmetric and positive definite. Hence, we have an invariant metric on the space of homogenous polynomials of degree m given by

$$g^{(m)}_{i,j} = \left(P^{(m)}_i, P^{(m)}_j\right) = \int_{\mathcal{G}} ([\mathcal{D}^m]^T(U)\mathcal{D}^m(U))_{ij}\sigma(U)dU. \quad (2)$$

With all the ingredients available, what remains is explicitly evaluating the invariant metric $g^{(m)}_{ij}$ given by the Eq. (2) for various values of n (the number of degrees of freedom) and m (the degree of the homogenous polynomials). These expressions are given below for a few values of m and n.

1. For $n = 2$ and $m = 2$, the matrix $g^{(m)}$ is a 10×10 matrix, whose diagonal entries are given by

$$g_{i,i} = \begin{cases} \frac{3}{2} & \text{for } i = 1,5,8,10; \\ \frac{8}{3} & \text{otherwise;} \end{cases}$$

while the non-zero non-diagonal entries are given by

$$g_{i,j} = \frac{1}{6} \quad \text{for } i,j = 1,5,8,10.$$

2. For $n = 2$ and $m = 3$, the matrix $g^{(m)}$ is a 20×20 matrix, whose diagonal elements are given by

$$g_{i,i} = \begin{cases} \frac{25}{24} & \text{for } i = 1,11,17,20; \\ \frac{9}{2} & \text{for } i = 6,7,9,15; \\ \frac{61}{24} & \text{otherwise;} \end{cases}$$

while the non-zero non-diagonal entries are given by

$$g_{i,j} = \begin{cases} & i,j = 1,5,8,10; \\ \dfrac{7}{24} \quad \text{for} & i,j = 2,11,14,16; \\ & i,j = 3,12,17,19; \\ & i,j = 4,13,18,20. \end{cases}$$

3. For $n = 3$ and $m = 2$, the matrix $g^{(m)}$ is a 21×21 matrix, whose diagonal elements are given by

$$g_{i,i} = \begin{cases} \frac{11}{8} & \text{for } i = 1,7,12,16,19,21; \\ \frac{5}{2} & \text{otherwise;} \end{cases}$$

while the non-zero non-diagonal entries are given by

$$g_{i,j} = \frac{1}{8} \quad \text{for } i,j = 1,7,12,16,19,21.$$

4. For $n = 3$ and $m = 3$, the matrix $g^{(m)}$ is a 56×56 matrix, whose diagonal elements are given by

$$g_{i,i} = \begin{cases} \frac{5}{6} & \text{for } i = 1,22,37,47,53,56; \\ \frac{21}{10} & \text{for } i = 2,3,4,5,6,7,12,16,19,21,23,24,25,26,27,31,34,36,38,39,40,41,44,46,48,49,50,52,54,55; \\ \frac{19}{5} & \text{otherwise;} \end{cases}$$

while the non-zero non-diagonal entries are given by

$$g_{i,j} = \begin{cases} & i,j = 1,7,12,16,19,21; \\ & i,j = 2,22,27,31,34,36; \\ \dfrac{1}{5} \quad \text{for} & i,j = 3,23,37,41,44,46; \\ & i,j = 4,24,38,47,50,52; \\ & i,j = 5,25,39,48,53,55; \\ & i,j = 6,26,40,49,54,56. \end{cases}$$

We now construct the invariant norm that quantifies the leading order nonlinearity making use of the metric $g^{(m)}$. As explained earlier, to ensure that $U(n)$ is the appropriate symmetry group, we first transform the symplectic map into its normal form and then write out the invariant norm for f_m in the transformed phase space coordinates. Consider the following definition of $I(z)$:

$$I(z) = \left(f_m^{tr}(z), f_m^{tr}(z)\right)^{1/2};$$

where $(\cdot,\cdot)$ is defined in Eq. (2). Using the expansion of $f_m^{tr}(z)$ in terms of the basis monomials, we obtain

$$\left(f_m^{tr}(z), f_m^{tr}(z)\right) = \left(\sum_{i=1}^{N(2n,m)} b_i P_i^{(m)}(z), \sum_{j=1}^{N(2n,m)} b_j P_j^{(m)}(z)\right) = \sum_{i=1}^{N(2n,m)} \sum_{j=1}^{N(2n,m)} b_i b_j \left(P_i^{(m)}(z), P_j^{(m)}(z)\right) = \sum_{i=1}^{N(2n,m)} \sum_{j=1}^{N(2n,m)} b_i b_j g_{ij}^{(m)}.$$

Since $(\cdot,\cdot)$ defined in Eq. (2) is a symmetric positive definite, bilinear form invariant under the action of $\mathcal{N}$, we have proved the following theorem.

Theorem 3. *The following definition of I(z) gives a norm that remains invariant under the action of U(n).*

$$I(z) = \left(\sum_{i=1}^{N(2n,m)} \sum_{j=1}^{N(2n,m)} b_i b_j g_{ij}^{(m)}\right)^{1/2}.$$

Notice that $I(z)$ is a function of the coefficients b_α which in turn are related to the coefficients a_α of the symplectic map $\mathcal{M}$ (and hence the original Hamiltonian system). Hence $I(z)$ is a function of the parameters s that we wish to optimize.

When $I(z)$ is a polynomial of degree 3, it quantifies the leading nonlinearity of the system. Therefore, one can attempt to vary one of the parameters describing the original system and minimize the norm $I(z)$ which serves as a "merit function". Once such a merit function is available for the Hamiltonian system, there already exists several papers [21–23,28,29] which provide extensive details regarding how to optimize the "performance" of a given system. Needless to say, the implementation requires a detailed knowledge about the system of interest so that it can be modeled by an appropriate Hamiltonian.

The norm can also be used to quantify the "closeness" of two nonlinear symplectic maps. Since the metric was defined to be invariant only under the action of the compact part of $Sp(2n,\mathbb{R})$, it (and consequently, the norm) still has some dependence on the linear part of the map. Therefore, it is meaningful only to compare maps that have the same linear part M. Let the two maps be defined as follows:

$$\mathscr{M} = \widehat{M}e^{\mathscr{L}_{f_3}}e^{\mathscr{L}_{f_4}}\cdots e^{\mathscr{L}_{f_n}}\cdots,$$
$$\mathscr{M}' = \widehat{M}e^{\mathscr{L}_{f'_3}}e^{\mathscr{L}_{f'_4}}\cdots e^{\mathscr{L}_{f'_n}}\cdots$$

Then the "distance" $d(\mathscr{M},\mathscr{M}')$ between the two maps $\mathscr{M}$ and $\mathscr{M}'$ (when restricted to degree 3) can be defined as follows

$$d(\mathscr{M},\mathscr{M}') = \|f_3 - f'_3\|.$$

References

[1] S. Blanes, Symplectic maps for approximating polynomial Hamiltonian systems, Phys. Rev. E 65 (2002) 056703.
[2] E. Chacon-Golcher, F. Neri, A symplectic integrator with arbitrary vector and scalar potentials, Phys. Lett. A 372 (2008) 4661–4666.
[3] S.A. Chin, S.R. Scuro, Exact evolution of time-reversible symplectic integrators and their phase errors for the harmonic oscillator, Phys. Lett. A 342 (2005) 397–403.
[4] J.F. Cornwel, Group Theory in Physics, vols. I and II, Academic Press, New York, 1984.
[5] V.V. Dodonov, Universal integrals of motion and universal invariants of quantum systems, J. Phys. A. Math. Gen. 33 (2000) 7721–7738.
[6] A.J. Dragt, Physics of high energy particle accelerators, in: R.A. Carrigan, F. Huson, M. Month (Eds.), AIP Conference Proceedings No. 87, American Institute of Physics, New York, 1982, pp. 147–313.
[7] A.J. Dragt, J.M. Finn, Lie series and invariant functions for analytic symplectic maps, J. Math. Phys. 17 (1976) 2215–2227.
[8] A.J. Dragt, E. Forest, K.B. Wolf, Foundations of a Lie algebraic theory of geometrical optics, in: J.S. Mondragon, K.B. Wolf (Eds.), Lie Methods in Optics, Springer-Verlag, Berlin, 1986, pp. 105–157.
[9] A.J. Dragt, F. Neri, G. Rangarajan, D.R. Douglas, L.M. Healy, R.D. Ryne, Lie algebraic treatment of linear and nonlinear beam dynamics, Ann. Rev. Nucl. Part. Sci. 38 (1988) 455–496.
[10] E. Forest, R.D. Ruth, Fourth-order symplectic integration, Physica D 43 (1990) 105–117.
[11] J. Gibbons, D.D. Holm, C. Tronci, Geometry of Vlasov kinetic moments: a bosonic Fock space for the symmetric Schouten bracket, Phys. Lett. A 372 (2008) 4184–4196.
[12] A. Giorgilli, A computer program for integrals of motion, Comp. Phys. Commun. 16 (1979) 331–343.
[13] S. Habib, R.D. Ryne, Symplectic calculation of Lyapunov exponents, Phys. Rev. Lett. 74 (1995) 70–73.
[14] J. Irwin, Using Lie algebraic maps for design and operation of linear colliders, Part. Accel. 54 (1996) 107–122.
[15] T.M. Janaki, G. Rangarajan, S. Habib, R.D. Ryne, Computation of Lyapunov spectrum for continuous time dynamical systems and discrete maps, Phys. Rev. E 60 (1999) 6614–6626.
[16] S.A. Khan, Wavelength-dependent modifications in Helmholtz optics, Int. J. Theor. Phys. 44 (2005) 95–125.
[17] V. Lakshminarayanan, R. Sridhar, R. Jagannathan, Lie algebraic treatment of dioptic power and optical aberrations, J. Opt. Soc. Am. A15 (1998) 2497–2503.
[18] H.E. Lomeli, Symplectic homogeneous diffeomorphisms, Cremona maps and the jolt representation, Nonlinearity 18 (2005) 1065–1071.
[19] R.I. McLachlan, H.S. Munthe-Kaas, G.R.W. Quispel, et al, Explicit volume-preserving splitting methods for linear and quadratic divergence-free vector fields, Found. Computat. Math. 8 (2008) 335–355.
[20] R.I. McLachlan, G.R.W. Quispel, et al, Explicit geometric integration of polynomial vector fields, BIT Numer. Math. 44 (2004) 515–538.
[21] D. Neuffer, Lumped correction of systematic multipoles in large synchrotrons, Part. Accel. 23 (1988) 21–35.
[22] D. Neuffer, Lumped correction of the multipole content in large synchrotrons, Nucl. Instrum. Meth. A 274 (1989) 400–403.
[23] D. Neuffer, E. Forest, A general formalism for quasi-local correction of multipole distortions in periodic transport systems, Phys. Lett. A 135 (1989) 197–201.
[24] A. Nijenhuis, H.S. Wilf, Computational Algorithms for Computers and Calculators, Academic Press, New York, 1978.
[25] M. Poźniak, K. Zyczkowski, M. Kuś, Composed ensembles of random unitary matrices, J. Phys. A: Math. Gen. 31 (1998) 1059–1071.
[26] G. Rangarajan, Symplectic completion of symplectic jets, J. Math. Phys. 37 (1996) 4514–4542.
[27] G. Ranagarajan, F. Neri, Kinematic moment invariants for linear Hamiltonian systems, Phys. Rev. Lett. 64 (1990) 1073–1075.
[28] G. Rangarajan, M. Sachidanand, Invariant metrics for nonlinear symplectic maps, Pramana - J. Phys. 58 (2002) 477–488.
[29] T. Sun, R. Talman, Numerical study of various lumped correction schemes for random multipole errors, SSC-N-500, 1988, unpublished.