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Generalized Replicator Dynamics as a Model of Specialization and Diversity in Societies

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ABSTRACT. We consider a generalization of replicator dynamics as a non-cooperative evolutionary game-theoretic model of a community of N agents. All agents update their individual mixed strategy profiles to increase their total payoff from the rest of the community. The properties of attractors in this dynamics are studied. Evidence is presented that under certain conditions the typical attractors of the system are corners of state space where each agent has specialized to a pure strategy, and/or the community exhibits diversity, i.e, all strategies are represented in the final states. The model suggests that new pure strategies whose payoff matrix elements satisfy suitable inequalities with respect to the existing ones can destabilize existing attractors if N is sufficiently large, and be regarded as innovations that enhance the diversity of the community.
KEYWORDS: Generalized replicator dynamics, evolutionary game theory, interacting agents, diversity, specialization.

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1. Introduction

Replicator dynamics [Hofbauer and Sigmund, 1988; Weibull, 1995, Samuelson, 1996; Vega-Redondo, 1996; and references therein] is a standard model in evolutionary biology describing the dynamics of growth and decay of a number of species under selection pressure. Formally, consider s species in an interactive environment, each genetically programmed to a particular behaviour with which it is identified. Tag the possible behaviours as 'strategies' belonging to the set $S = \{1, 2, \ldots, s\}$ in the sense of game theory. Let $A = [[a_{ij}]]$, $1 \leq i, j \leq s$, denote the payoff matrix with the following interpretation: the i-th species (playing strategy i) receives a payoff ($\approx$ contribution to its fitness) of a_{ij} on interaction with species j. The rate of increase of the population $y_i(t)$ of species i at time t is assumed to be proportional to $y_i(t)$ and the birth rate minus the death rate. The latter is assumed constant, the former a constant modulated by an additive term equal to the average (current) payoff to i. Letting $x_i(t) = y_i(t)/(\sum_j y_j(t))$ denote the population share of i, simple algebra (see, e.g., Weibull, 1995) leads to replicator dynamics

$$\dot{x}_i(t) = x_i(t)[\sum_j a_{ij}x_j(t) - \sum_{k,j} x_k(t)a_{kj}x_j(t)]. \tag{1.1}$$

The description makes the underlying biological formalism clear. A species is not allowed to change its strategy or 'genetic traits'. That is, 'ontogenetic adaptation' whereby individuals adapt to environment in their lifetime, is not permitted. Eq. (1.1) stands for 'phylogenetic adaptation' by the species as a whole in response to birth-death mechanism modulated by selection pressure. This makes eminent biological sense, because ontogenetic adaptation is not inherited, phylogenetic is.

More recently, however, the model has crossed the boundaries of biology. It has been taken up by economists as a model of economic learning. The motivation comes from the problem of equilibrium selection in game theory. Having concurred that Nash equilibrium is the most reasonable equilibrium concept for noncooperative games, one is confronted with the problem of choosing a 'natural' one from the overabundance thereof in a typical scenario. Many refinements were proposed, culminating in an elaborate theory of equilibrium selection [Harsanyi and Selten, 1988]. A recent departure from this trend has been to leave the static equilibrium framework in favour of dynamic models of disequilibrium and look for their asymptotic equilibria. Replicator dynamics and its variants have been a prominent front runner in this activity [see e.g. Mailath, 1992; Fudenberg and Levine, 1995]. The attractive feature of this model of evolutionary economics is that it builds in bounded rationality (agents are not perfect optimizers with perfect knowledge, infinite memory and computational ability, but make simple decisions based on immediate payoffs) and inertia (the adaptation takes place in incremental steps). Though originally meant to model 'phylogenetic learning' in biology, economists have successfully recovered it as a limiting case of models of individual, or ontogenetic learning [Borgers and Sarin, 1993; Ritzberger and Weibull, 1995] thus deriving 'macrobehaviour' from 'micromotives' [Schelling, 1978]. Prompted by this, there have been alternative models of evolutionary learning in economics [Young, 1993; Ellison and Fudenberg, 1995, for example].

Another motivation for considering dynamic models comes from the study of complex adaptive systems. A major question in this area is how do complex organizations emerge spontaneously in certain systems (e.g., chemical, biological and social systems). Implicit in the question is a notion of time evolution. One is interested the mechanisms whereby, starting from initial conditions where there is no organizational complexity, the system can end up in a state where an organizational character is discernible in which specific, nonrandom interaction between components of the system resulting in an overall coherence has emerged. In this context generalizations of replicator dynamics have been considered in the literature (see Stadler, Fontana and Miller, 1993, and references therein).

In this paper we discuss another generalization [Borkar, Jain and Rangarajan, 1997], motivated by somewhat different concerns than earlier works. Mathematically, our model is simply a multipopulation variant of replicator dynamics [Weibull, 1994, Chapter 5]. Here, N distinct populations, each with its strategy profile, evolve under selection pressure. Let $p_i^\alpha(t)$ denote the fraction of α-th population that plays strategy i. The dynamics then is

$$\dot{p}_i^\alpha(t) = p_i^\alpha(t)\left[\sum_{\beta\neq\alpha}\sum_j a_{ij}p_j^\beta(t) - \sum_{\beta\neq\alpha}\sum_{k,j} p_k^\alpha(t)a_{kj}p_j^\beta(t)\right],\ 1\le\alpha\le N,\ 1\le i\le s. \tag{1.2}$$

Our interpretation, however, is different. We view Eq. (1.2) as a model of ontogenetic learning wherein the superscript α, $1 \le \alpha \le N$, stands for α-th agent, $p_i^\alpha(t)$ being the probability with which she plays strategy i at time t. Thus $p^\alpha(t) \equiv (p_1^\alpha(t), p_2^\alpha(t), \dots, p_s^\alpha(t))^T$ is her 'mixed strategy' at time t and $\sum_{\beta\neq\alpha}\sum_{k,j} p_k^\alpha(t)a_{kj}p_j^\beta(t)$ represents the payoff received by α from the rest of the community at time t. Eq. (1.2) then shows how agents adapt their strategies in response to payoffs received. The interpretation of the r.h.s. of Eq. (1.2) is that each agent increases the weight of those pure strategies that do better than her current (mixed) strategy in the environment of the current strategies of other agents, and decreases the weight of those that do worse, in proportion to the difference. The community as a whole evolves as all the individuals simultaneously make their updates. To emphasize this shift in paradigm from populations to individuals, we dub Eq. (1.2) 'generalized replicator dynamics' or GRD, rather than 'multipopulation replicator dynamics'. For contrast, Eq. (1.1) will be called 'pure replicator dynamics' or PRD.

Not only our interpretation, but the focus of our study is also different from that of Weibull [1994, Chapter 5]. Specifically, our primary aim will be to identify and analyse two key phenomena, which we dub 'specialization' and 'diversification'. By 'specialization', we refer to the situation in which every agent plays a pure strategy, i.e., for each α, the probability vector $p^\alpha(t)$ is concentrated at a single point of the strategy space. In other words, while having the choice of choosing from the set of mixed strategies, thereby apportioning her resources (energy, time etc) on more than one activity, she chooses to put them all into a single activity. The kind of situations we are interested in are those in which this phenomenon goes hand in hand with another, that of 'diversification'. We say that diversification occurs in a community if for each possible strategy choice, there is at least one agent at any time who opts for it with positive probability.

That is, no available strategy is completely discarded by the community as a whole. Note that specialization is not a prerequisite for diversification. We do not require that agents in a diversified society use pure strategies in order for it to be considered as such. In fact, we do not even insist that they necessarily play distinct strategies. The society would still be deemed as being diversified if all of them played the same mixed strategy that puts positive probability mass on every strategy option. We do, however, look for situations where both specialization and diversification occur at the same time: while each agent puts all her eggs in one basket, no basket is left empty. Needless to say, we expect such behaviour only when the payoff matrix A satisfies additional constraints. For example, if a strategy choice is strictly dominated by every other, one would indeed expect the society as a whole to opt out of it. Thus a significant part of the effort will be directed towards precisely nailing down what those conditions on A might be.

The motivation for such an exercise is almost self-evident. We wish to capture qualitatively a phenomenon that recurs across a variety of evolutionary paradigms. If one looks at the evolution of human society from a primitive hunter-gatherer society to a modern town, one sees progressive specialization of essential tasks among individuals while no essential task is being completely discarded by the society as a whole. If one looks at the evolution of species on earth, one sees genetic strands diversifying to fill in specific ecological niches till their common ancestry is no longer obvious. In economics, the evolution from old manufacturing and trading patterns to a modern corporation is marked by extreme specialization and decentralization of tasks, while the corporation as a whole concurrently does a multitude of tasks that go into the meeting of its final objective.

This is just by the way of motivation. We delegate the issues of interpretation of the mathematical model to the discussion at the end of the paper and return to the mathematical formalism per se.

The quantity $\bar{x}_i \equiv (1/N)\sum_{\alpha=1}^{N} p_i^\alpha$ is the population average of the probability that the strategy i is being played by the entire collection of agents and can be interpreted as the relative weight of the i^{th} strategy in the population. One might wonder whether this averaged out quantity satisfies the PRD equation when the individual p_i^α satisfy GRD. The answer is no even in the limit of large number of agents, as will be discussed later. Nevertheless, we find that in the large N limit GRD contains a fair amount of "memory" about some of the PRD structures, in particular the interior equilibrium point of PRD, as we shall note later. The important point being made here, however, is that although our concept of diversification bears some similarity with concepts such as persistence and permanence for the PRD [Hofbauer and Sigmund, 1988], it is not subsumed or implied by these.

This paper is organized as follows: Section 2 reviews some known properties of PRD and GRD attractors, sets the notation and outlines the similarities and differences between PRD and GRD. This is followed by further results on the non-corner attractors of GRD. Section 3 identifies the conditions for the existence of maximal diversity in GRD, or preservation of all strategies in the population. Here we note the importance of large N in promoting diversity in GRD. Explicit sufficient conditions for $s = 2$ and 3 are given, as well as partial results for higher s. Section 4 studies corner solutions in GRD which correspond to specialization in the community. Some theorems concerning the coexistence of specialization and diversity in the community are proved. Finally in section 5 we summarize our results from the previous sections, discuss their possible implications concerning specialization, diversity and innovation in economic communities, and mention some future directions.

2. NON-CORNER ATTRACTORS OF GRD

2.1. NON-CORNER EQUILIBRIA OF PRD

In this subsection we recall some well known facts about PRD (see Hofbauer and Sigmund, 1988, chapters 13, 19) that will be useful later and set some notation. Let J denote the simplex of s-dimensional probability vectors, $J = \{\mathbf{x} = (x_1, \cdots, x_s)^T \in \mathbf{R}^s | x_i \geq 0,\ \sum_{i=1}^s x_i = 1\}$. J is the full configuration space of PRD dynamics, and is invariant under it, i.e., an s-dimensional probability vector remains so under evolution given by (1.1). Let L denote a subset of S with at most $s-1$ elements. Then the set $F_L = \{\mathbf{x} \in J | x_k = 0, k \in L\}$ is called a face of J. In particular if $L = \{k\}$, the corresponding face $F_{\{k\}}$ (denoted F_k for brevity) is such that on this face the k^{th} strategy is not represented in the population; the full diversity of available strategies is lost. All the faces F_k together constitute the boundary of J. The relative boundary of each F_k in turn is a union of the 'subfaces' $F_{kj} \equiv F_{\{k,j\}} = F_k \cap F_j$, $\partial F_k = \cup_{j \neq k} F_{kj}$, and so on. Finally, if L has exactly $s-1$ elements, say $L = S - \{i\}$, the corresponding face F_L is a singleton, and will be called the 'corner' C_i of J; only strategy i survives in the population at C_i. All faces, subfaces, corners are invariant under PRD; if $x_k = 0$ at some time, it remains so thereafter. It follows that corners are trivially equilibrium points.

We are interested in the interior of J, i.e., $J^\circ = J - \cup_{k=1}^s F_k$. (The superscript $^\circ$ for a set will always denote its relative interior.) For trajectories that remain in J°, all strategies are always present, since $0 < x_i < 1$ for all i. A point $\mathbf{x} \in J^\circ$ that is a stationary point of (1.1) will be called an interior equilibrium point (IEP) of PRD. $\mathbf{x}$ is an IEP of PRD only if the $s+1$-dimensional vector $\mathbf{X} = (x_0, \mathbf{x})$ satisfies

$$B\mathbf{X} = \mathbf{E_0}, \tag{2.1}$$

where B is the $s+1$-dimensional matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & & & & \\ -1 & & A & & \\ \vdots & & & & \\ -1 & & & & \end{pmatrix} \tag{2.2}$$

and $\mathbf{E_0}$ is the $s+1$-dimensional unit vector $(1, 0, 0, \cdots, 0)^T$. (In (2.1) $\mathbf{X}$ is represented as an $s+1$-dimensional column vector). To see this, note that the 0^{th} component of (2.1) is just the normalization condition $\sum_{i=1}^s x_i = 1$, and the i^{th} component is the equation $\sum_{j=1}^s a_{ij}x_j - x_0 = 0$. The latter statement means that the payoff for all strategies is x_0, independent of strategy, which, together with the normalization condition is equivalent to the vanishing of the bracket on the r.h.s. of (1.1). The solution to (2.1) is

$$x_i = {B^{-1}}_{i0} = u_i/\det B, \quad u_i \equiv \text{Cofactor of } B_{0i}, \tag{2.3}$$

$$x_0 = {B^{-1}}_{00} = \det A/\det B, \tag{2.4}$$

provided $\det B \neq 0$. Note from the structure of B that

$$\det B = \sum_{i=1}^{s} u_i. \tag{2.5}$$

x_i being probabilities must be greater than or equal to zero, and for an interior point, must be positive. This is guaranteed from (2.3) and (2.5) if

A1: $u_i \neq 0 \;\; \forall \;\; i$, and all u_i have the same sign.

A1 is a necessary and sufficient condition (on the payoff matrix A) for PRD to have an isolated IEP, which, if it exists, is unique and given by (2.3). Sufficiency is evident from the above. To see that **A1** is also necessary, observe that $\det B = 0$ would imply that the solution of the linear equation (2.1) is not isolated, since we could always add to $\mathbf{X}$ a vector proportional to a zero eigenvector of B.

Dynamics within F_k can be studied using the submatrix $A^{(k,k)}$ of the full payoff matrix A ($M^{(p,q)}$ denotes the matrix obtained from a matrix M by deleting its p^{th} row and q^{th} column; we will follow the convention for labelling the indices that if the indices of M go over a set T, then the row indices of $M^{(p,q)}$ will go over the set $T - \{p\}$, and the column indices over the set $T - \{q\}$). Denote $u_i^{(k)} \equiv$ Cofactor of$(B^{(k,k)})_{0i}$, $i \in S - \{k\}$, and define the following condition:

A1$_k$: $u_i^{(k)} \neq 0 \;\; \forall \;\; i \neq k$, and all $u_i^{(k)}$ have the same sign.

It is clear from the above that if **A1**$_k$ is satisfied, then the relative interior of F_k, $F_k^\circ = F_k - \partial F_k$ has an equilibrium point denoted $\mathbf{x}^{(k)} = (x_1^{(k)}, \ldots, x_s^{(k)})$ with

$$x_i^{(k)} = \frac{u_i^{(k)}}{\det B^{(k,k)}}, \quad i \neq k, \quad x_k^{(k)} = 0,. \tag{2.6}$$

with the payoff to all agents at this equilibrium point given by

$$x_0^{(k)} = \frac{\det A^{(k,k)}}{\det B^{(k,k)}}. \tag{2.7}$$

A1$_k$ is sufficient but not a necessary condition for F_k° to have an equilibrium point. An equilibrium point can also arise if $u_k = 0$ and all other u_i are non-vanishing and have the same sign. However the latter equilibrium point is non-generic, i.e., it happens only when the components of A satisfy an equation ($u_k = 0$). This situation is structurally unstable: it can be destroyed by a small perturbation of A. Thus generically, F_k° has at most one equilibrium point (one, given by (2.6) if **A1**$_k$ holds, and none if it is violated). The generic equilibrium point is characterized by equal payoffs to all strategies except k, which is just ignored, and the non-generic one by the fact that even the k^{th} strategy receives the same payoff as all other strategies. An analogous situation holds for the equilibrium points in the interior of subfaces F_{kj}, etc.

For all trajectories that do not converge to the boundary of J, the barycentre of every invariant probability measure supported in the interior is the IEP of PRD. To see this, define $\bar{x}_i = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, x_i(t)$ and $c = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \sum_{k,j=1}^{s} x_k(t)a_{ij}x_j(t)$ along some common convergent subsequence. From (1.1) it follows that $\dot{x}_i/x_i = (Ax)_i - x^T Ax$. Integrating this w.r.t. t from zero to T, dividing by T and taking the limit along appropriate subsequence gives

$$\lim_{T\to\infty} \frac{1}{T}[\ln x_i(T) - \ln x_i(0)] = (A\bar{x})_i - c. \tag{2.8}$$

Since by assumption x_i is bounded away from zero, the l.h.s. vanishes. Therefore $(A\bar{x})_i$ is independent of i, which, together with the normalization condition on $\bar{x}$ means that $\bar{x}_i$ must coincide with the IEP (2.3). Similarly, for generic A, for all trajectories in F_k that do not converge to its relative boundary, the barycentre of every invariant probability measure supported in F_k° is given by (2.6).

In the remainder of this section we will show that above results for PRD have appropriate generalizations in GRD, and also discuss some departures.

2.2. GRD PRELIMINARIES

2.2.1. NOTATION, DEFINITION OF SPECIALIZATION AND DIVERSITY

The configuration space of GRD will be denoted $J^N = \Pi_{\alpha=1}^N J^{(\alpha)}$ where $J^{(\alpha)}$ is a copy of J for the α^{th} agent. A point of J^N will be denoted $p = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^N)$, where $\mathbf{p}^\alpha = (p_1^\alpha, p_2^\alpha, \ldots, p_s^\alpha) \in J^{(\alpha)}$. For many purposes it will be convenient to think of J^N to be embedded in $\mathbf{R}^{Ns} = \Pi_{\alpha=1}^N \mathbf{R}^s(\alpha)$ where $\mathbf{R}^s(\alpha)$ is a copy of $\mathbf{R}^s$ for the α^{th} agent and $J^\alpha \subset \mathbf{R}^s(\alpha)$. Then we can write $p = (p_1, p_2, \ldots, p_{Ns}) \equiv (p_b)$, $p_b \geq 0$. b runs over the index set $S_N = \{1, 2, \ldots, Ns\} = \cup_{\alpha=1}^N S(\alpha)$, where the set $S(\alpha) = \{(\alpha-1)s+1, (\alpha-1)s+2, \ldots, \alpha s\}$ will be called the α^{th} block (of indices). $\hat{e}_b$ will denote the unit vectors along the axes of R^{Ns}, the d^{th} component of $\hat{e}^b$ is $(\hat{e}^b)_d = \delta_d^b$. Thus $J^N = \{p \in \mathbf{R}^{Ns} |\ p_b \geq 0,\ \sum_{b\in S(\alpha)} p_b = 1\ \ \forall\ \ \alpha = 1, 2, \ldots, N\}$, and $J^{(\alpha)} = \{p \in \mathbf{R}^{Ns} |\ p_a \geq 0\ \forall a,\ p_b = 0,\ b \notin S(\alpha),\ \sum_{b\in S(\alpha)} p_b = 1\}$. The notation $b = (\alpha, i)$ will denote that b is the i^{th} element in the α^{th} block.

We now define a face of J^N. Let L be any subset of S_N such that it contains at most $s-1$ indices from every block. Then the set $F_L = \{p \in J^N | p_b = 0,\ b \in L\}$ will be called a face of J^N. A face $F_{L'}$ will be called a 'subface' of F_L if L is a proper subset of L'. Clearly if $L = \emptyset$, $F_L = J^N$. If L is of order $N(s-1)$, i.e., it has exactly $s-1$ elements from each block, then F_L is a singleton and will be called a corner of J^N. If for some α, $\mathbf{p}^\alpha$ goes to a corner C_i of $J^{(\alpha)}$ ($p_j^\alpha = \delta_{ij}$), this of course means that the agent α has opted for the pure strategy i, and we say that α has 'specialized' in strategy i. At a corner of J^N, every agent has specialized to some strategy or the other and we say that the community is 'fully specialized'. J^N is invariant under GRD, as the L^1 norm and positivity of every $\mathbf{p}^\alpha$ is preserved under (1.2). Every face of J^N is invariant under GRD, as $p_i^\alpha(t) = 0\ \forall\ t$ if it is zero for some t. Therefore corners are trivially equilibrium points. This does not, however, mean that they are stable. If corners of J^N are the only asymptotically stable attractors of the dynamics defined by (1.2), we will say that the latter exhibits ***specialization***. For then, the community will always end up being fully specialized, starting from (a small perturbation of) any initial condition.

If $L = \{k, s+k, 2s+k, \dots, (N-1)s+k\}$ for some $k \in S$ the corresponding face F_L is one where every agent has opted out of the k^{th} strategy. Such faces will be particularly important for us and we will denote them as F_k. They are analogues of the F_k defined in the previous subsection, where a particular strategy k becomes extinct from the population. At a face F_k the full diversity of strategies is lost. Conversely the complement of the set $\cup_{k=1}^{s} F_k$ in J^N corresponds to the situation where every strategy is represented in the community and we denote this set by J_D^N, with the subscript D denoting the fact that the full 'diversity' of strategies is maintained. Note that we use the word 'diversity' not to signify the variation between individual agents, but as a measure of the size of set of strategies being pursued. Indeed we can have no variation but full diversity if all agents pursue the same mixed strategy: for all α, $\mathbf{p}^\alpha = \mathbf{c} \in J^\circ$. When $\mathbf{p}^\alpha$ is independent of α, the community is completely 'homogeneous' since all agents are doing the same thing. Also we remark that the community can be fully specialized and diversified at the same time: each agent chooses a pure strategy and every strategy is chosen by some agent or the other.

In order to maintain the full diversity of strategies in the society, no strategy k should be dynamically opted out of by all agents. That is, no trajectory that starts in the interior of J^N should end up in any of the faces F_k. Better still, any trajectory in a face F_k should become unstable with respect perturbations that take it into the interior of J^N. The basic idea is that while under the dynamics (1.2) a trajectory in F_k will always remain in it, in practice in a real society, agents will always be subject to influences that draw their attention to strategies that do not exist in the population, but are valid strategies within the capabilities of the agents. Such influences can be thought of as perturbing the society away from the face F_k. If F_k is unstable to small perturbations, then this perturbation will grow. If all faces F_k are unstable, no strategy can in practice become extinct from the population. We will say that the dynamics (1.2) exhibits ***diversity*** if every trajectory in each of the faces F_k of J^N becomes unstable at some time or the other with respect to small perturbations that deform it away from F_k. This notion of diversity is analogous to the notions of permanence [Schuster, Sigmund and Wolff, 1979], persistence [Butler, Freedman and Waltman, 1986, and references therein], etc., which have been defined in PRD to study the extinction of strategies (see Hofbauer and Sigmund, 1988 for definitions).

It is evident that the boundary of J^N is given by the union of all faces in which exactly one of the p_b's is zero, $\partial J^N = \cup_{b=1}^{Ns} F_{\{b\}}$. In general the relative boundary of F_L is given by $\partial F_L = \cup_{b \in \bar{L}} \{p \in F_L | p_b = 0\}$, where $\bar{L}$ is the complement of L in S_N. The relative interior of a face F_L is defined as $F_L^\circ = F_L - \partial F_L$. Note that unlike in PRD here the boundary of J^N not the same as $\cup_{k=1}^{N} F_k$; the latter is a subset of ∂J^N. Similarly $\cup_{j \neq k} F_{kj}$ (where $F_{kj} = F_k \cap F_j$) is a subset of the boundary of F_k.

In general, $p \in J^N$ is an equilibrium point for Eq. (1.2) if for every $b = (\alpha, i) \in S_N$, either $p_b = 0$, or,

$$\sum_{\beta \neq \alpha} \sum_{j} a_{ij} p_j^\beta = \sum_{\beta \neq \alpha} \sum_{k,j} p_k^\alpha a_{kj} p_j^\beta. \tag{2.9}$$

Note that the right hand side of the above equation is independent of i. Combining this with the fact that $(p_1^\alpha, p_2^\alpha, \cdots, p_s^\alpha)$ is a probability vector, one can replace Eq. (2.9) by: $\sum_{\beta \neq \alpha} \sum_j a_{ij} p_j^\beta$ is the same for all i for which $p_i^\alpha > 0$. At an equilibrium point an agent receives the same payoff for all the pure strategies she has not opted out of.

2.2.2. Review of some known results for GRD

We now recall some known facts about equilibria (and other attractors) of GRD from Chapter 5, Weibull (1995). (See this reference for relevant definitions.)

Theorem 2.1 If strategy i is either dominated or iteratively dominated, $p_i^\alpha(t) \to 0 \ \forall \alpha$.

Theorem 2.2 All interior equilibria and Liapunov stable equilibria of Eq. (1.2) are Nash equilibria. Also, if an interior trajectory of Eq. (1.2) converges to a point, the same must be a Nash equilibrium.

Theorem 2.3 Any compact set in the relative interior of a face cannot be asymptotically stable.

Note that strict Nash equilibria are perforce pure strategy Nash equilibria and therefore correspond to corner solutions which we identify as 'specialization' in behavioural sense.

2.2.3. Differences between PRD and GRD

Note that (1.1) is of the form $\dot{x}_i = x_i(f_i(\mathbf{x}) - \bar{f}(\mathbf{x}))$, with $f_i(\mathbf{x}) = \sum_{i \in S} a_{ij} x_j$ and $\bar{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{f}(\mathbf{x})$. The GRD equation (1.2) does not have quite this form. Let $\bar{A}$ denote the $Ns \times Ns$ matrix given by

$$\begin{pmatrix} 0 & A & A & \cdots & A \\ A & 0 & A & \cdots & A \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A & A & \cdots & \cdots & 0 \end{pmatrix} \tag{2.10}$$

where 0 is a $s \times s$ zero matrix. Then indeed (1.2) can be written as $\dot{p}_b = p_b(f_b(p) - \bar{f}(p))$, with $f_b(p) = \sum_{c \in S_N} \bar{A}_{bc} p_c$. But now $\bar{f}(p)$ is no longer given by $p^T f(p)$, but by $p^{(\alpha_b)^T} f(p)$, where $p^{(\alpha_b)}$ is the projection of p onto the block α_b to which b belongs ($p_c^{(\alpha_b)} = p_c$ if c is in the same block as b and zero otherwise).

Second, note that in GRD every agent plays against other agents only, not against herself. That is, if agent α were to play pure strategy i, her payoff,

$\sum_{\beta \neq \alpha} \sum_j a_{ij} p_j^\beta$, excludes the term $\beta = \alpha$. We will see later that this fact is crucial for the coexistence of specialization and diversity.

If the initial point of a trajectory in GRD is homogeneous, the trajectory remains homogeneous for all time, and evolves according to (1.1) except that the time is speeded up by a factor of $N-1$. This is evident upon substituting $p_i^\alpha = p_i^\beta$ in (1.2). Thus in the 'homogenous sector', GRD is the same as PRD.

The sum $\bar{x}_i \equiv (1/N)\sum_{\alpha=1}^{N} p_i^\alpha$ equals the average probability that strategy i is being played in the entire community, and is therefore the analogue of x_i in PRD. We can ask how $\bar{x}_i$ evolves in GRD. It is easy to see that

$$\dot{\bar{x}}_i = N[\bar{x}_i \sum_j a_{ij}\bar{x}_j - \sum_{k,j} x_{ik} a_{kj} \bar{x}_j - \frac{1}{N}\sum_j x_{ij} a_{ij} + \frac{1}{N}\sum_{k,j} x_{ikj} a_{kj}], \qquad (2.11)$$

where $x_{ik} \equiv \frac{1}{N}\sum_\alpha p_i^\alpha p_k^\alpha$ and $x_{ikj} \equiv \frac{1}{N}\sum_\alpha p_i^\alpha p_k^\alpha p_j^\alpha$. The r.h.s. of (2.11) is not proportional to the r.h.s. of (1.1), except for homogeneous trajectories in which case $x_{ik} = \bar{x}_i \bar{x}_k$, $x_{ikj} = \bar{x}_i \bar{x}_k \bar{x}_j$. Thus in general $\bar{x}_i$ does not follow the PRD. One might have hoped that when the number of agents N is large $\bar{x}_i$ follows PRD, but even that is not the case due to variation among the agents. For example, consider an asymptotically stable equilibrium point, which we are assured by theorem 2.3 must be a corner, in which the number of agents pursuing the pure strategy i is given by n_i. Then the difference between x_{ik} and $\bar{x}_i \bar{x}_k$ is $n_i(N\delta_{ik} - n_k)/N^2$, which can be comparable to the former two even for large N (except for homogeneous corners).

One of our main results in this paper is that even though variation among agents, which is generic in GRD causes the evolution of $\bar{x}_i$ to be different from PRD, under suitable conditions for A and sufficiently large N, $\bar{x}_i$ nevertheless converges to the IEP of PRD given by (2.3).

2.3. Some results on non-corner attractors of GRD

We shall be interested in 'generic' behaviour, i.e., behaviour that holds true for A in an open and dense set of $\mathbf{R}^{s \times s}$. Thus all claims made in this subsection are for generic A, not necessarily for all A. First, we prove that

Theorem 2.4 There exists at most one isolated equilibrium in the interior of J^N. It exists if and only if **A1** is satisfied and then it is homogeneous (all agents pursue the same mixed strategy), and coincides with the IEP of PRD, $p_i^\alpha = x_i \ \forall\ \alpha, i$ (Borkar, Jain and Rangarajan, 1998).

Proof: Let $p \in J^{N^\circ}$ be an equilibrium point. By definition no p_b is zero in the interior of J^N. Therefore (2.9) holds for all α, i. Define $x_0^\alpha \equiv \sum_{\beta\neq\alpha}\sum_{i,j} p_i^\alpha a_{ij} p_j^\beta$, and $v_i^\alpha \equiv \sum_{\beta \neq \alpha} p_i^\beta$. Then $\sum_{i=1}^{s} v_i^\alpha = N-1 \ \forall\ \alpha$, and (2.9) can be written as $B\mathbf{X}^\alpha = (N-1)\mathbf{E}_0$ for every α, where $\mathbf{X}^\alpha \equiv (x_0^\alpha, v_1^\alpha, v_2^\alpha, \ldots, v_s^\alpha)^T$. This equation is the same as (2.1) except for the factor of $N-1$ on the right. Therefore it has a solution if and only if **A1** holds. The solution is unique and given by $v_i^\alpha = (N-1)x_i \ \forall\ \alpha$. Since $p_i^\alpha - p_i^\beta = v_i^\beta - v_i^\alpha = 0$, it follows that the equilibrium point is homogeneous and given by $p_i^\alpha = x_i$. □

Corollary 2.1 For generic A there exists at most one equilibrium point of GRD in the interior of face F_k. This generic equilibrium point exists if $\mathbf{A1}_k$ is satisfied and then it is homogeneous and is given by $p_i^\alpha = x_i^{(k)}\ \forall\ \alpha, i$, where the r.h.s. is given by (2.6), and the payoff to each agent at this point is $(N-1)x_0^{(k)}$, with $x_0^{(k)}$ given by (2.7). If $\mathbf{A1}_k$ is violated, there is generically no equilibrium point in F_k°.

This follows from arguments similar to those discussed above for the faces F_k of PRD. □

In fact, one can show that generically, there is at most one equilibrium of Eq. (1.2) in the relative interior of any face of J^N. Note that (2.9) can be written as

$$(\bar{A}p)_b = p^{(\alpha_b)^T}\bar{A}p, \tag{2.12}$$

where we use the notation $b = (\alpha_b, i_b)$. The r.h.s. depends only on α_b and not on i_b.

Let us consider an equilibrium point in the interior of the face F_L. By definition, if $p \in F_L$, $p_b = 0$ except for $b \in L^C \equiv H$. Let H have m elements. Then $m \geq N$ because $H = L^C$ implies that H contains at least one element from each block. If we partition H as $H = \bigcup_{\alpha=1}^N H_\alpha$, where H_α is the intersection of H with the α^{th} block, then $H_\alpha \neq \emptyset\ \ \forall\ \ \alpha$. For the m components of p which are non-zero, the condition for equilibrium is (2.9) or (2.12). Therefore p is an equilibrium point in F_L° if $(\bar{A}p)_b$ is independent of $i_b\ \forall\ \ b \in H$ (or equivalently, if $(\bar{A}p)_b = (\bar{A}p)_{b'}$ whenever b and b' are in the same block and are elements of H).

This condition can be interpreted geometrically as follows: Define $\hat{E}$ as the subspace of R^{Ns} consisting of all vectors $r = \sum_{b=1}^{Ns} r_b\hat{e}^b$ such that $r_b = r_{b'}$ whenever b and b' are in the same block and are elements of H. There are $m-N$ conditions on the components of r, hence $\hat{E}$ is $Ns - (m-N) = N(s+1) - m$ dimensional. Define $D = \{q \in R^{Ns} | q_b = \sum_d \bar{A}_{bd}p_d,\ p \in F_L\}$. It is evident from the conditions mentioned above that if $p \in F_L^\circ$ is an equilibrium point of GRD then $q = \bar{A}p$ belongs to $\hat{E}$, i.e., $\hat{E} \cap D^\circ \neq \emptyset$, and vice versa.

Note that since the only nonzero components p_d appearing in the definition of D are those for which $d \in H$, only the corresponding columns $\bar{A}_d$ of $\bar{A}$ appear. Then, interpreting each column $\bar{A}_d$ of $\bar{A}$ as a point in R^{Ns}, D is the compact convex subset of the polytope generated by the m points $N\bar{A}_d$, $d \in H$. D is contained in an affine space of dimension at most $m-N$. To see this let $\hat{D}$ denote the smallest affine space (i.e., translate of a subspace of $\mathbf{R}^{Ns}$) that contains D. Since the p_d's in the definition of D must add to 1 when summed over d belonging to any block, and there are N such conditions, the dimension of F_L is $m-N$, and that of $\hat{D}$ is $m-N$ if the $\bar{A}_d$'s are linearly independent, and less if not.

Theorem 2.5 For generic A, there is at most one equilibrium of GRD per face of J^N.

Proof: The proof is based on the correspondence discussed above between the equilibrium points in the interior of the face F_L and the convex set $\hat{E} \bigcap D^\circ$. Without loss of generality we ignore the point faces (corners). Suppose $\hat{E} \bigcap D^\circ$ has dimension one or more. We must show that this situation is structurally unstable under a perturbation of A. Clearly, $m > N$, or else D itself is a singleton. In particular, at least one H_α, say H_{α_0}, has more than one element. If $\hat{E} \bigcap D^\circ$ has dimension one or more, it contains a line segment. Every point on a line segment is a convex combination of its endpoints. Thus there are two or more $\bar{A}_d$'s whose uncountably many convex combinations lie in $\hat{E}$. From our definition of $\hat{E}$, all of them must have the property that their components corresponding to indices in H_{α_0} are equal. It is clear that, if for two distinct vectors, infinitely many convex combinations of their prescribed components are equal to one another, the vectors themselves must have this property. Then the $\bar{A}_d$'s in question must also have this property. But this property is destroyed by a suitable perturbation of A.

Now suppose $\hat{E} \bigcap D = \emptyset$. Since $\hat{E}$, D are closed convex and D compact, this is equivalent to a nonzero distance between the two and therefore stable under a small perturbation of A, hence of D. Let $\hat{E} \bigcap D^\circ = \{y\}$. Since $\hat{D}$, $\hat{E}$ have at most complementary dimensions in $\mathbf{R}^{Ns}$, this can happen only when $\hat{E}$, D are transversal at y, a situation stable under a small perturbation of A, hence of D. Finally, if $\hat{E} \bigcap D = \{y\}$ with y in the relative boundary of D, a small perturbation of A will lead to either a similar situation, or to one of the scenarios considered above. The claim follows. □

In the remainder of this section, we assume A to be such that the above conclusions hold. The next two results relate general (non point) limit sets of GRD with its equilibria. Recall that a trajectory $x(\cdot)$ in a Euclidean space is said to converge to a point $\bar{x}$ (or a set C) in Cesaro sense if $\frac{1}{t}\int_0^t x(r)dr \to \bar{x}$ (resp. C) as $t \to \infty$. For the remainder of this subsection, by convergence we will mean convergence in the Cesaro sense. This is to take care of some pathological behaviour that can occur.

As before let F_L be a face whose relative interior is $F_L^\circ = \{p \in J^N | p_b > 0 \text{ iff } b \in \bar{L}\}$. Define H, D, $\hat{E}$ as before.

Theorem 2.6 Any trajectory $p(\cdot)$ of GRD in F_L° converges either to its relative boundary or to an equilibrium point in F_L°.

Proof Suppose it does not converge to the relative boundary of F_L. Then there exist $t_k \uparrow \infty$ such that $\{p_b(t_k)\}$ remains bounded away from zero for each $b \in H$. Define $\bar{p}(t) \in F_L$ by

$$\bar{p}_b(t) = \frac{1}{t}\int_0^t p_b(r)dr \quad \forall\, b, t. \tag{2.13}$$

By dropping to an appropriate subsequence, let

$$\bar{p}(t_k) \to p^* \in F_L \tag{2.14}$$

and

$$\frac{1}{t_k}\int_0^{t_k} p^{(\alpha_b)}(t)^T \bar{A} p(t)dt \to c^{\alpha_b} \in R \tag{2.15}$$

where $p^{(\alpha)}$ is the projection of p onto J^α. Rewrite GRD as

$$\dot{p}_b(t)/p_b(t) = (\bar{A}p(t))_b - p^{(\alpha_b)}(t)^T \bar{A} p(t), \quad b \in H. \tag{2.16}$$

Integrating, we have, for $b \in H$,

$$\frac{\ln p_b(t_k)}{t_k} - \frac{\ln p_b(0)}{t_k} = \frac{1}{t_k}\int_0^{t_k} [(\bar{A}p(r))_b - p^{(\alpha_b)}(r)^T \bar{A}p(r)]dr. \tag{2.17}$$

Let $k \to \infty$ to obtain

$$(\bar{A}p^*)_b = c^{\alpha_b}, \quad b \in H. \tag{2.18}$$

Now $\bar{A}p^* \in D$ while an Ns-vector whose b^{th} element is c^{α_b}, independent of i_b if $b \in H$, zero otherwise, is clearly in $\hat{E}$. By the preceding theorem and our choice of A ('generic'), Eq. (2.18) uniquely specifies p^* as the unique GRD equilibrium in F_L°. Now, $t \to \bar{p}(t)$ is continuous and hence converges to a compact connected set C in F_L. The foregoing implies that $C \bigcap F_L^\circ = p^*$. By connectedness of C, $C = \{p^*\}$. This completes the proof. □

We now give an alternative interpretation to this result. GRD defines a compact flow in J^N and has a nonempty compact convex subset of invariant probability measures whose extreme points correspond to the ergodic measures [Sell, 1971, Ch. IX]. Recall that the barycentre of a probability measure on a Euclidean space $\mathbf{R}^n$ is the point $x_0 \in \mathbf{R}^n$ for which $f(x_0) = \int f d\mu$ for all affine (i.e. linear plus constant) f. The above theorem can then be rephrased as

Theorem 2.6′ Every invariant probabality measure of GRD supported in F_L, whose support intersects F_L°, has a common barycentre given by the unique GRD equilibrium in F_L°. □

Corollary 2.2 If **A1** holds, all trajectories that do not converge to the boundary of J^N converge (in the Cesaro sense) to $p_i^\alpha = x_i \ \forall\ \alpha, i$ with x_i given by (2.3). Also if $\mathbf{A1}_k$ holds, all trajectories in the face F_k that do not converge to its boundary converge to $p_i^\alpha = x_i^{(k)} \ \forall\ \alpha, i$, with $x_i^{(k)}$ given by (2.6).

This follows upon using theorem 2.4 and corollary 2.1 in theorem 2.6′. □

We also have:

Corollory 2.3 If F_L° does not have any GRD equilibria, every trajectory of GRD therein must converge to the relative boundary of F_L. □

Thus the typical picture is: each non-point face has zero or one GRD equilibrium. In the former case, all trajectories initiated in the face must converge to its relative boundary (in the Cesaro sense). In the latter, the equilibrium is not asymptotically stable, though it can still be Liapunov stable. Any trajectory in the interior of the face must converge either to this equilibrium or to the relative boundary of the face. The former allows, e.g., a 'centre', i.e., a family of periodic orbits around an equilibrium. If the condition **A1** holds, there is a unique equilibrium point in the interior of J^N which is homogeneous and coincides with the IEP of PRD for every agent. For every trajectory in J^N that does not converge to its boundary, the time average of the trajectory (over long times) converges to this point. A similar result holds for the face F_k (at which the k^{th} strategy becomes extinct) if condition $\mathbf{A1}_k$ is satisfied.

Finally, we discuss the presence of 'heteroclinic cycles' in our dynamics. A heteroclinic cycle for an ordinary differential equation consists of finitely many equilibria, each one connected to the next by a trajectory of the o.d.e which has the former as its α-limit point and the latter as its ω-limit point. The following two theorems are useful in making the connection between heteroclinic cycles and GRD dynamics:

Theorem 2.7 For generic payoff matrices, every asymptotically stable attractor for the GRD contains one or more corner equilibria.

Proof Let C be an asymptotically stable attractor of the GRD which is not an equilibrium point. Then C is a compact subset of the state space. Let F be an (open) face and G its boundary such that $F \cup G$ contains C. Let F be k-dimensional. Then G is a union of faces of dimension $k-1$ or less. C cannot be in the relative interior of F, because then it cannot be asymptotically stable. Thus $C' = C \cap G$ is nonempty. G itself is invariant for the o.d.e. If C' is not asymptotically stable for the flow restricted to G, nor will be C. Thus C' is asymptotically stable for the flow restricted to G and by the same argument as before, it cannot be in the union of relative interiors of $(k-1)$-dimensional faces of G. That is, it has a nonempty intersection with the complement thereof in G, which is a union of faces of dimension $k-2$ and lower. This argument can be repeated to conclude that C must contain some corner equilibria.□

Theorem 2.8 If an attractor contains only finitely many ω- and α- limit points (which then must be equilibria), it is a heteroclinic cycle.

This is same as Corollary 1.4, pp. 146 in [Benaim and Hirsch, 1996].

The two theorems together strongly suggest that we can expect heteroclinic cycles in GRD. In fact, in our numerical simulations, in the few cases that did not lead to convergence to corner equilibria, the behaviour was strongly suggestive of heteroclinic cycles as attractors. That is, the trajectory spent ever increasing amounts of times near a succession of corner equilibria, moving relatively quickly, when it did, between two. We are tempted to conjecture that convergence to corner equilibria or heteroclinic cycles containing them is generic. We remark that long periods of stasis punctuated by rapid change have also been noted in other evolutionary contexts.

3. CONDITIONS FOR DIVERSITY IN GRD

As defined in section **II B**, the property of diversity has to do with the instability of trajectories in F_k with respect to perturbations away from F_k. Consider a generic point $p \in F_k$. By definition $p_k^\alpha = 0 \ \forall \ \alpha$, or $p_b = 0 \ \forall \ b \in \{k, s+k, 2s+k, \ldots, (N-1)s+k\}$. Now consider a perturbation of this point that takes it away from F_k. This is achieved by making $x^\alpha \equiv p_k^\alpha$ nonzero. Consider the dynamics of the N-tuple $x = (x^1, \ldots, x^N)$ under GRD (x is just the collection of the k^{th} components of the $\mathbf{p}^\alpha$), treating the other components of p (which determine the location on F_k) as fixed parameters. The linearized $N \times N$ jacobian matrix of this dynamics evaluated turns out to be diagonal and therefore its eigenvalues at $x = 0$ will determine the stability of the point $p \in F_k$

with respect to perturbations away from F_k. (Strictly speaking, we are not justified in treating the other components of p as fixed parameters because every $\mathbf{p}^\alpha$ is normalized to unity and a nonzero x^α implies that the other components of $\mathbf{p}^\alpha$ must also change. However, we will show later that this does not affect the Jacobian matrix.)

We use the notation that the capital Roman indices I, J, K, etc. go over the set $S - \{k\}$. Then $\sum_{j=1}^{s} a_{ij}p_j^\beta = \sum_J a_{iJ}p_J^\beta + a_{ik}x^\beta$. Thus (1.2) gives

$$\dot{x}^\alpha = f^\alpha(x;p) = x^\alpha[\sum_{\beta\neq\alpha}\sum_J a_{kJ}p_J^\beta - \sum_{\beta\neq\alpha}\sum_{K,J} p_K^\alpha a_{KJ}p_J^\beta \\ +\text{terms linear and quadratic in } x]. \tag{3.1}$$

Therefore the above mentioned jacobian matrix is

$$J_{\alpha\gamma}(p) \equiv \frac{\partial f^\alpha(x;p)}{\partial x^\gamma}\Big|_{x=0} = \delta_{\alpha,\gamma}[\sum_{\beta\neq\alpha}\sum_J a_{kJ}p_J^\beta - \sum_{\beta\neq\alpha}\sum_{K,J} p_K^\alpha a_{KJ}p_J^\beta]. \tag{3.2}$$

Since this is already diagonal, its eigenvalues as a function of the point p in the face F_k are

$$\lambda_\alpha(p) = \sum_{\beta\neq\alpha}\sum_J a_{kJ}p_J^\beta - \sum_{\beta\neq\alpha}\sum_{K,J} p_K^\alpha a_{KJ}p_J^\beta. \tag{3.3}$$

Now we discuss the justification of treating x's as variables independent of p. Note that at $x = 0$ the terms linear and quadratic in x inside the [ ] in $f^\alpha(x;p)$ do not contribute. In general we could consider stability along any trajectory in J^N that is transversal to F_k at p. For small x, such a trajectory could be parametrized as $p_K^\alpha(t) = p_K^\alpha + C_K^\alpha t$, $x^\alpha(t) = C^\alpha t$, where the C's are constants that can be chosen suitably to preserve the normalization of $\mathbf{p}^\alpha$. Effectively, this means that for small x, we can replace $\mathbf{p}^\alpha$ in $f(x;p)$ by a linear function of x^α. This makes no contribution to the [ ], and hence the jacobian, at $x = 0$. Thus the stability along any trajectory will be governed by the λ_α given by (3.3). Thus we have proved

Lemma 3.1 A trajectory in F_k passing through p is unstable with respect to perturbations away from F_k at p if the largest of the $\lambda_\alpha(p)$ given by (3.3), $\alpha = 1, 2, \ldots, N$, is greater than zero at p. □

λ_α has a simple economic interpretation. The first term is the payoff to agent α if she were to pursue the pure strategy k at p, other agents remaining at p. The second term is the payoff she is actually getting at p, where she is playing $\mathbf{p}^\alpha$. The instability of F_k if $\lambda_\alpha > 0$ is just the statement that movement towards the k^{th} strategy yields a better payoff to agent α than her current strategy.

3.0.1. Stability of interior trajectories of F_k

We now wish to consider the long time average of $\lambda_\alpha(p)$ as p moves around in F_k, i.e., the quantity $\bar{\lambda}_\alpha = \frac{1}{T}\int_0^T dt\ \lambda_\alpha(p(t))$, for trajectories in F_k that do not converge to its relative boundary. Recall from corollary 2.3 that such trajectories exist only if the interior of F_k has an equilibrium point and generically this happens only if condition $\mathbf{A1}_k$ holds (corollary 2.1). As a particular case of the analysis in the proof of theorem 2.6 and theorem 2.6′, it then follows that $\bar{\lambda}_\alpha$ is obtained by substituting for p in (3.3) the (generically unique) equilibrium point of F_k given by corollary 2.1. Thus

$$\bar{\lambda}_\alpha = \frac{(N-1)}{\det B^{(k,k)}}[\sum_J a_{kJ}u_J^{(k)} - \det A^{(k,k)}]. \tag{3.4}$$

We now show that the quantity in [ ] in the last equation is just $-u_k$. For ease of writing determinants, we consider the case $k = s$.

$$\begin{aligned}
-u_s = \quad & -(-1)^s \begin{vmatrix} -1 & a_{11} & a_{12} & \cdots & a_{1,s-1} \\ -1 & a_{21} & a_{22} & \cdots & a_{2,s-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & a_{s-1,1} & a_{s-1,2} & \cdots & a_{s-1,s-1} \\ -1 & a_{s1} & a_{s2} & \cdots & a_{s,s-1} \end{vmatrix} \\
&= (-1)^{s-1} \begin{vmatrix} -1 & & & & \\ -1 & & & & \\ \vdots & & A^{(s,s)} & & \\ -1 & & & & \\ -1 & a_{s1} & a_{s2} & \cdots & a_{s,s-1} \end{vmatrix} \\
&= \begin{vmatrix} -1 & a_{s1} & a_{s2} & \cdots & a_{s,s-1} \\ -1 & & & & \\ -1 & & & & \\ \vdots & & A^{(s,s)} & & \\ -1 & & & & \\ -1 & & & & \end{vmatrix} \\
&= -\det A^{(s,s)} - a_{s1} \begin{vmatrix} -1 & \\ -1 & \\ \vdots & (A^{(s,s)})^{(\cdot,1)} \\ -1 & \\ -1 & \end{vmatrix} \\
&\quad + a_{s2} \begin{vmatrix} -1 & \\ -1 & \\ \vdots & (A^{(s,s)})^{(\cdot,2)} \\ -1 & \\ -1 & \end{vmatrix} \\
&\quad - \ldots (-1)^{s-1} a_{s,s-1} \begin{vmatrix} -1 & \\ -1 & \\ \vdots & (A^{(s,s)})^{(\cdot,s-1)} \\ -1 & \\ -1 & \end{vmatrix} \\
&= -\det A^{(s,s)} + \textstyle\sum_{J=1}^{s-1} a_{sJ}u_J^s.
\end{aligned} \tag{3.5}$$

Here we have used the notation that if M is an $n \times n$ matrix then $M^{(\cdot,q)}$ is an $n \times (n-1)$ matrix obtained from M by removing its q^{th} column. In the last line we have used the expression

$$u_J^{(s)} = \text{Cofactor of}(B_{0J}^{(s,s)}) = (-1)^J \begin{vmatrix} -1 & \\ -1 & \\ \vdots & (A^{(s,s)})^{(\cdot,J)} \\ -1 & \\ -1 & \end{vmatrix} . \tag{3.6}$$

Therefore

$$\bar{\lambda}_\alpha = -\frac{(N-1)}{\det B^{(k,k)}} u_k \equiv \bar{\lambda}. \tag{3.7}$$

Since $\bar{\lambda}$ is a time average along a trajectory, if it is positive it means that $\bar{\lambda}_\alpha(t)$ must have been positive for a sufficiently large subset of the trajectory. At all these times the trajectory must therefore have been unstable with respect to perturbations that take it away from F_k. Therefore we have proved

Theorem 3.1 All trajectories in the face F_k that do not converge to its relative boundary are unstable with respect to perturbations that take them away from F_k if the r.h.s. of (3.7) is greater than zero. □

This theorem covers periodic cycles and other attractors of F_k that do not converge to its relative boundary.

Note that $\bar{\lambda}_\alpha$ is independent of α and is expressed completely in terms of the payoff matrix. It also coincides with the transversal eigenvalue at the equilibrium point of F_k° in PRD (compare with section 19.5 of Hofbauer and Sigmund, 1988). This is because the equilibrium point of F_k° in GRD is homogeneous and given by corollary 2.1.

3.0.2. Stability of the corners of F_k

By definition, at every corner of J^N each agent pursues a pure strategy. Note that two agents using the same strategy are indistinguishable otherwise. Thus the key feature of a corner is the s-vector of non-negative integers $\mathbf{n} = (n_1, \ldots, n_s)$ where n_i is the number of agents using strategy i in the given corner equilibrium, $1 \le i \le s$ (thus $\sum_i n_i = N$). Two corner equilibria with same associated $\mathbf{n}$ vector are interchangeable, since they differ only in the identity of the agents, irrelevant for our purposes. On F_k, $n_k = 0$. On such a corner $\mathbf{n}$, $\lambda_\alpha(\mathbf{n})$ depends on α only through I, the strategy pursued by the α^{th} agent. It is easy to see that

$$\lambda_\alpha(\mathbf{n}) = P_k - P_I - h_{kI}, \tag{3.8}$$

where $P_j \equiv \sum_{i=1}^s a_{ji} n_i$ and $h_{ij} \equiv a_{ij} - a_{jj}$. There are thus at most $s-1$ distinct eigenvalues at a corner of F_k. The condition for it to be unstable with respect to perturbations away from F_k is that the largest of them is greater than or equal to zero.

We are interested in conditions on A such that *all* corners of F_k are unstable. For this purpose we will now study in detail the cases $s = 2, 3$. Before turning to this analysis, note the following point: If a constant is added to all elements of a column of A, the r.h.s. of (1.1) and (1.2) is unchanged because of cancellations between the two terms. Thus one can shift each column of A by a constant without affecting the dynamics. By choosing the constant in the j^{th} column to be a_{jj} we arrive at the matrix $A' = (h_{ij}) = (a_{ij} - a_{jj})$ whose diagonal entries are zero. All conclusions about PRD or GRD are unaltered whether one works with the payoff matrix A or A'. This includes the location and stability of equilibria.

Introduce the following

Definition: A is diagonally subdominant if $a_{ii} < a_{ji}\ \forall\ j \neq i,\ \{i,j\} \subset S$.

That is, $h_{ij} > 0\ \forall\ i \neq j$. Note that the PRD with the matrix A' is then called a 'catalytic network' (see Section 20.4, Hofbauer and Sigmund, 1988).

s=2

In this case F_1 and F_2 are themselves both corners of J^N. There is one eigenvalue λ at each corner, and from (3.8) it follows that for F_1 the eigenvalue is $\lambda = (N-1)h_{12}$ and for F_2, $\lambda = (N-1)h_{21}$.

Therefore if A satisfies the condition

A2: A is diagonally subdominant,

then for $s = 2$, F_1 and F_2 are both unstable for a community of two or more agents. The condition **A2** will appear repeatedly in the subsequent analysis. Note that $u_1 = -h_{12}$, $u_2 = -h_{21}$, and $\det B = -(h_{12} + h_{21})$ are all negative if **A2** holds, hence diagonal subdominance implies **A1** or the existence of an equilibrium point in the interior of J^N. This equilibrium point is given (theorem 2.1) by $\mathbf{p}^\alpha = \frac{1}{h_{12}+h_{21}}(h_{12}, h_{21})$ for all α, and by theorem 2.6$'$, the trajectories $p(t)$ that do not converge to the boundary of J^N converge to this point in the Cesaro sense.

s=3

For $s = 3$, **A2** no longer implies **A1**; the latter is an independent condition.

Lemma 3.2 For $s = 3$, if both **A1** and **A2** hold, then all trajectories in the interior of each F_k that do not converge to its boundary are unstable. Further, there exists a positive number N_0 depending on A such that for $N > N_0$ all corners of F_1, F_2, F_3 are also unstable (Borkar, Jain and Rangarajan, 1998).

The first part follows from the analysis in Gadgil, Nanjundiah and Gadgil, 1980, section 5, where it is shown that under **A1**,**A2** for $s = 3$ $\bar{\lambda}_\alpha$ is positive. Therefore the condition for theorem 3.1 is satisfied. (See also Hofbauer, Schuster, which states that for $s = 3$, catalytic networks satisfying **A1** are permanent. Permanence of PRD implies the positivity of transversal eigenvalues $\bar{\lambda}_\alpha$.) For completeness we give a proof here, which will also be useful in the sequel. Under **A1** and **A2** the sign of the u_i is determined by the sign of det A' (this is true for any s). To see this consider the representation A' so that the payoff $x'_0 = \sum_{i,j} x_i h_{ij} x_j$ to each pure strategy at the IEP of PRD is, by **A2**, a sum of positive terms. Since from (2.4) $x'_0 = \det A' / \det B'$, this means $\operatorname{sgn}\det B' = \operatorname{sgn}\det A'$. Now $\det B' = \det B$ for any s. This is evident upon adding a_{jj} times the first column of B to its j^{th} column for every j, whereby in (2.2) A gets converted to A' without change of determinant on both sides. But from **A1** and (2.5) all u_i have the same sign as $\det B$, hence all u_i have the same sign as $\det A'$. For $s = 3$, $\det A' = h_{12}h_{23}h_{31} + h_{13}h_{32}h_{21}$ is manifestly positive under **A2** and hence $\det B$ and u_1, u_2, u_3 are all positive. Also, since each $A^{(k,k)}$ is a 2×2 diagonally subdominant matrix, from the remarks above for $s = 2$, it is evident that $\det B^{(k,k)} < 0$ for each k. Thus $\bar{\lambda}_\alpha > 0$ for every k and theorem 3.1 implies that all trajectories in each F_k that do not converge to its relative boundary are unstable with respect to perturbations away from F_k.

Now consider the stability of corner equilibrium points of F_k. For concreteness consider F_3. Corners of F_3 are of two types,

Case 1: only one strategy survives at the corner. Then $\mathbf{n} = (N, 0, 0)$ or $(0, N, 0)$. In the former case (3.8) implies $\lambda = (N-1)h_{31}$ and in the latter case $\lambda = (N-1)h_{32}$. By **A2** both corners are unstable.

Case 2: Both strategies 1 and 2 survive at the corner of F_3. Then $\mathbf{n} = (n_1, n_2, 0)$ with both n_1 and n_2 positive integers and $n_1 + n_2 = N$. There are then two eigenvalues from (3.8), $\lambda_1 = h_{31}n_1 + h_{32}n_2 - h_{12}n_2 - h_{31}$, and $\lambda_2 = h_{31}n_1 + h_{32}n_2 - h_{21}n_1 - h_{32}$. Let us assume that this corner is stable, hence both λ_1, λ_2 are negative. The condition $\lambda_1 < 0$ (upon eliminating $n_1 = N - n_2$) reduces to $(h_{12} + h_{31} - h_{32})n_2 > (N-1)h_{31}$. Since $n_2, N-1, h_{31}$ are all positive this means that the combination $h_{12} + h_{31} - h_{32}$ is also positive, and

$$(N-1)\frac{h_{31}}{h_{12} + h_{31} - h_{32}} < n_2. \tag{3.9}$$

Similarly $\lambda_2 < 0$ implies that $h_{21} + h_{32} - h_{31}$ is positive (as can be seen by eliminating n_2) and further,

$$n_2 < \frac{(N-1)(h_{21} - h_{31})}{h_{21} + h_{32} - h_{31}} + 1. \tag{3.10}$$

Combining the two, we get

$$(N-1)\frac{h_{31}}{h_{12} + h_{31} - h_{32}} < \frac{(N-1)(h_{21} - h_{31})}{h_{21} + h_{32} - h_{31}} + 1, \tag{3.11}$$

which can be rearranged into the form

$$(N-1)[-h_{21}h_{12} + h_{21}h_{32} + h_{31}h_{12}] < (h_{12} + h_{31} - h_{32})(h_{21} + h_{32} - h_{31}). \tag{3.12}$$

But the quantity in [ ] on l.h.s. of this inequality is just u_3 (as evaluated from the definition (2.3)), which is positive. Thus we have

$$N < \frac{(h_{12} + h_{31} - h_{32})(h_{21} + h_{32} - h_{31})}{u_3} + 1. \tag{3.13}$$

Note that the r.h.s. is a function of A alone and is finite, say $N_0(A)$. If N is chosen larger than $N_0(A)$, this inequality is violated. That is, for $N > N_0(A)$, the corner of F_3 under consideration cannot be stable. We have thus proved that under **A1**,**A2**, all corners of F_3 are unstable for $N > N_0(A)$. Similarly one may