

Kinematical conservation laws applied to study geometrical shapes of a solitary wave

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Abstract

Kinematical conservation laws (KCL), giving the successive positions of a moving curve in a plane, is used to describe all possible geometrical shapes of the crest-line (the line joining the highest points) of a curved solitary wave on a shallow water. The KCL is an under-determined system of two equations. We assume that the length of the curved solitary wave in the direction transverse to the direction of its propagation is very large compared to a length measuring the breadth of the solitary wave. This allows us to treat a section of the solitary wave by a plane perpendicular to the crest-line to be an one dimensional solitary wave and helps us to find an additional relation to close the KCL and solve the problem completely.

1. Introduction

Considerable amount of research has been done on the stability (both longitudinal and transverse) of solitary waves in the last 30 years (Bridges, 2001). In this paper, we have considered geometrical shapes of stable curved solitary waves in shallow water of constant depth. By a curved solitary wave, we mean a wave whose extent in the transverse direction (i.e., a direction perpendicular to the direction of propagation) is very large compared to its extent in normal direction (i.e., the direction of propagation) and the shape of the wave in a normal section is locally a solitary wave. This allows us to define at any time a crest-line Ω_t , the locus of the highest points of the local solitary waves in the normal sections. Our aim in this paper is to study the successive positions of Ω_t and its geometric shape starting from its initial configuration. When the amplitude of the wave on the crest-line vary slowly (compared to the variation of the amplitude in the local solitary wave), transverse waves are induced which propagate along the crest-line. The slow amplitude variation is coupled to the variation in the normal direction of the crest-line.

Propagation of waves (or shock waves) along the crest-line have been discussed by Ostrovsky and Shrira (1976), Miles (1977), Shrira (1980), Zakharov (1986) and Pederson (1994), some of these use Whitham's equations. Quite exhaustive results have been obtained by these authors. But one needs to fit each kink (shock - as usually called by these authors) on the crest-line individually since Whitham's equations are not in conservation form. Thus, long term solution of the problem becomes quite cumbersome. We shall write the basic equations for the propagation of the crest-line in physically realistic conservation forms, which are true for any propagating curve Ω_t in a plane. These are kinematical conservation laws (KCL), first derived by Morton, Prasad and Ravindran, 1992, for an appropriately defined velocity m of Ω_t , the Mach number of curve and angle θ which normal to Ω_t makes with the x -axis in terms of an appropriately defined ray

coordinate system (ξ, τ) . Here $\tau = \text{constant}$ give the successive positions of Ω_t (or Ω_τ) and $\xi = \text{constant}$ are rays - in this case orthogonal to Ω_t . Let g be the metric associated with the variable ξ , then the KCL are

$$(g \sin \theta)_\tau + (m \cos \theta)_\xi = 0, \quad (g \cos \theta)_\tau - (m \sin \theta)_\xi = 0 \quad (1.1)$$

Whitham's equations follow from these equations. The system (1.1) is an under-determined system of equations for three dependent variables, the dynamics of the propagating curve appears in additional partial differential equations (Monica and Prasad, 2001) or an additional relation $g = g(m)$, where g is a known function. We shall use the local solitary wave solution to determine g in the form

$$g(m) = (m - 1)^{-3/2} e^{-(3/2)(m-1)} \quad (1.2)$$

For $0 < m - 1 \ll 1$, to which our theory applies $g(m) = (m - 1)^{-3/2}$, which is equivalent to the closure relation used by Miles (see his equations (2.3a,b)) and Ostrovskii and Shrira (1976). Miles deduced this relation from the expression for the solution of solitary wave in water wave problem in a channel of slowly varying breadth. We deduce (1.2) by a method which will be applicable to solitary waves in all physical systems in which an equation for curved solitary waves can be obtained.

Use of KCL provides a new understanding of the phenomenon of kinks - it shows that the kinks are basically geometric shocks in a ray coordinate system. KCL makes computation of the successive positions of the crest-line not only very easy but very robust - the method of numerical solution can be continued for a very long time even if there are more than one kink, which interact among themselves and also with more complicated solutions than simple wave solutions. This is because, the whole range of sophisticated methods of numerical solution of hyperbolic conservation laws (Prasad and Sangeeta 1999, and Monica and Prasad 2001) are applicable to KCL.

Baskar and Prasad (2001) have recently worked out existence and uniqueness of the Riemann problem for KCL with a general form of the metric function $g = G(m)$ and discussed all possible geometric shapes of the propagating curve Ω_t . We note that the function (1.2) satisfies their assumptions on $G(m)$. Thus, their results are applicable to the crest-line of a curved solitary waves in a shallow water.

2. Multi-dimensional KdV equation

First we briefly mention various length and time scales and non-dimensional quantities involved in discussion of waves in shallow water.

H = a length scale characterizing the depth of the undisturbed water,

λ = wave length of the waves,

A = a measure of the maximum height of the wave, it has the dimension of length.

$$\epsilon = (H/\lambda)^2 \quad , \quad \delta = A/H \quad , \quad \bar{h} = Hh \quad , \quad \bar{\eta} = H\delta\eta \quad (2.1)$$

$$\left. \begin{aligned} (\bar{x}, \bar{y}, \bar{z}) &= (\lambda x, \lambda y, Hz) \quad , \quad \bar{t} = \epsilon^{-1/2} (H/g)^{1/2} t \\ (\bar{u}, \bar{v}, \bar{w}) &= \delta (gH)^{1/2} (u, v, \epsilon^{1/2} w) \quad , \quad \bar{p} = \bar{\rho} g H p \end{aligned} \right\} \quad (2.2)$$

where $(\bar{u}, \bar{v}, \bar{w})$ are components of the velocity in $\bar{x}, \bar{y}, \bar{z}$ directions, the undisturbed free surface is $\bar{z} = 0$, $\bar{z} = \bar{\eta}(\bar{x}, \bar{y})$ is the disturb free surface, \bar{h} is the constant depth, \bar{p} is the excess pressure above the atmospheric pressure and $\bar{\rho}$ is the constant density.

The water wave is dispersive, but in the long wave limit $\epsilon \rightarrow 0$ (without small amplitude assumption), it is well known that it supports non-dispersive waves and is governed

by a system of hyperbolic equations (e.g., see Prasad and Ravindran, 1977, equations (2.28) - (2.30)). These are equations in variables η, \tilde{u} and \tilde{v} where \tilde{u} and \tilde{v} are defined as velocity components in x, y directions respectively upto order ϵ terms but after carefully removing z -dependent parts from the velocity components. This system supports nonlinear nondispersive curved waves with eigenvalues

$$c_1 = \sqrt{h + \delta\eta} - \delta(n_1\tilde{u} + n_2\tilde{v}), \quad c_2 = \sqrt{h + \delta\eta}, \quad c_3 = \sqrt{h + \delta\eta} + \delta(n_1\tilde{u} + n_2\tilde{v}) \quad (2.3)$$

where $n_1 = \cos\theta$ and $n_2 = \sin\theta$ are components of the unit normal to the wavefront. The system can have simple wave solution of third family propagating in a fixed direction (n_1, n_2) (section 3.1.3, Prasad 2001). The simple wave is a nonstationary phenomenon, it cannot reduce to a steady solution in any frame of reference. The stationary nature of a solitary wave results from the balance of the nonlinear effects in the simple wave by dispersion. The balance is beautifully described by the KdV equation under small amplitude assumption, however this balance takes place in certain situations - in one case when

$$\delta = \epsilon^{1/3} = \epsilon_1 \quad (2.4)$$

where ϵ_1 is a short length scale characterizing a neighbourhood of the wavefront over which the disturbance is concentrated (see Prasad and Ravindran, 1977, result (3.17)). Derivation of the KdV equation requires that in the small amplitude perturbations ($\delta \ll 1$) $\delta\eta, \delta\tilde{u}$ and $\delta\tilde{v}$ are related by

$$\tilde{u} = n_1\eta/\sqrt{h}, \quad \tilde{v} = n_2\eta/\sqrt{h} \quad (2.5)$$

As is well known, the solitary wave is a solution of the KdV equation. The most important point to note is that the velocity of propagation of the solitary wave is not given by the eigenvalue c_3 .

The plane KdV equation with wavefront normal in (n_1, n_2) -direction can be modified by taking (n_1, n_2) to vary slowly over a length scale L such that

$$H \ll \lambda \ll L \quad (2.6)$$

Let $\phi(x, y, t)$ is the phase function such that $\phi = 0$ is the curved wavefront, and approximate the equations of surface water waves in propagation space (i.e., equations (2.31) - (2.33) of Prasad and Ravindran, 1977) in an ϵ_1 neighbourhood of $\phi = 0$. This leads to the modified KdV equation for the propagation of curved waves on a shallow water from the equation (4.5) of Prasad and Ravindran (1977) in the form

$$\frac{d\eta}{dt} - \sqrt{h}\eta\Omega + \frac{h}{6} \{(\phi_x^2 + \phi_y^2)\}^{3/2} \eta_{sss} = 0 \quad (2.7)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\sqrt{h} + \frac{3}{2\sqrt{h}}\delta\eta \right) \left(n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} \right) \quad (2.8)$$

$$\Omega = -\frac{1}{2} \left(\frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y} \right) \quad (2.9)$$

and s is the fast variable defined by

$$s = \phi(x, y, t)/\epsilon \quad (2.10)$$

Note that we do not wish to replace ϕ by the linear phase function ϕ_0 as done in Prasad and Ravindran (1977) and do not wish to go up to the equation (4.17) there.

The function ϕ satisfies the eikonal equation $\phi_t + \left(\sqrt{h} + \frac{3}{2\sqrt{h}}\delta\eta \right) |\nabla\phi| = 0, \nabla =$

$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $(n_1, n_2) = \nabla\phi/|\nabla\phi|$. This means that the derivative $\frac{d}{dt}$ is along the paths given by the corresponding ray equations,

$$\frac{dx_\alpha}{dt} = \left(\sqrt{h} + \frac{3\delta\eta}{2\sqrt{h}}\right)n_\alpha, \quad \frac{dn_\alpha}{dt} = -\frac{3\delta}{2\sqrt{h}}L\eta \quad (2.11)$$

where

$$L = \nabla - \mathbf{n}\langle\mathbf{n}, \nabla\rangle. \quad (2.12)$$

The equations (2.7), and (2.11) are coupled together. Unlike the purely nondispersive problem (where the last term in (2.7) is absent) as in chapter 6 Prasad (2001), numerical solution of the system (2.7), (2.11) and (2.12) appears to be extremely complex. However, unlike the well known KP (Kadomstev and Petvaishvili) equation this system is a true generalization of the KdV equation to multi-dimensions for the propagation of a curved dispersive waves in high frequency approximation. Our aim in this paper is to study propagation not of a general dispersive wave but only a curved solitary wave. In the next section, we shall use the equation (2.7) to determine the function g which appears in (1.2).

3. Equation for the average flux of energy

Consider now a point on the crest-line of a curved solitary wave such that at $t = 0$, the point coincides with a point $P(x_p, y_p)$. In a two-dimensional neighbourhood N_p in (x, y) -plane, of P of linear dimension ϵ_1 , we can write $\phi = n_1(x - x_p) + n_2(y - y_p) - \sqrt{ht} + O(\epsilon_1^2)$. Let $\tilde{x} = (x - x_p)/\epsilon_1$, $\tilde{y} = (y - y_p)/\epsilon_1$, then \tilde{x} and \tilde{y} are of order one in N_p . Then $\tilde{\Omega} = -\frac{1}{2}\left(\frac{\partial n_1}{\partial \tilde{x}} + \frac{\partial n_2}{\partial \tilde{y}}\right) = -\frac{\epsilon}{2}\left(\frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y}\right) = \epsilon\Omega$, where Ω is of order one. Hence, when we consider the equation (2.7) locally in the neighbourhood N_p with

$$s = \tilde{s} = \{n_1(x - x_p) + n_2(y - y_p) - \sqrt{ht}\}/\epsilon_1 \quad (3.1)$$

the curvature term can be neglected and it reduces locally to the KdV equation

$$\eta_{t'} + \frac{3}{2\sqrt{h}}\eta\eta_{\tilde{s}} + \frac{h}{6}\eta_{\tilde{s}\tilde{s}\tilde{s}} = 0 \quad (3.2)$$

where

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \sqrt{h}\left(n_1\frac{\partial}{\partial x} + n_2\frac{\partial}{\partial y}\right) \quad (3.3)$$

The local solitary wave satisfying (3.2) is

$$\eta_0 = 3c\alpha \sec h^2\left(\frac{\sqrt{c}}{2}(s - \beta ct)/\beta\right) \quad (3.4)$$

where

$$\alpha = 2^{2/3}3^{-4/3}h^{4/3}, \quad \beta = 6^{-1/3}h^{5/6} \quad (3.5)$$

(3.4) gives the amplitude A of the solitary wave $A = \delta \max_{\mathbb{R}} \eta_0 \equiv \delta\eta_c = 3c\alpha\delta$. Its normal velocity C (which is also the velocity of the crest-line), is given by $s - \beta ct = \text{constant}$ i.e., $C = \sqrt{h} + \beta c\delta$. We define the Mach number of the crest-line as $m = C/\sqrt{h}$ and hence the relation between m and the amplitude A at the point P is

$$m - 1 = \frac{\beta\delta_c}{\sqrt{h}} = \frac{\beta A}{3\alpha\sqrt{h}} = \frac{A}{2h} \quad (3.6)$$

As mentioned after the relation (2.5), the most important influence of the balance between the nonlinearity and dispersion in maintenance of the solitary wave shape is that the crest-line velocity is not the same as eigenvalue $c_3 = \sqrt{h} + \{3/(2\sqrt{h})\}\delta\eta_c$ but it is equal to

$$C = \sqrt{h} + \{\beta/(3\alpha)\}\delta\eta_c = \sqrt{h} + \left\{1/(2\sqrt{2})\right\}\delta\eta_c \quad (3.7)$$

which results in the relation (3.6).

The ray velocity of the solitary wave (i.e., also that of the crest-line) is $C(\cos\theta, \sin\theta)$. Hence the time rate of change, $\frac{d}{dT}$, when we move with the crest-line is

$$\frac{d}{dT} = \frac{\partial}{\partial t} + C \left(n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} \right) \quad (3.8)$$

Rearranging terms in the multi-dimensional KdV equation (2.7), we get

$$\begin{aligned} \frac{d\eta}{dT} - \sqrt{h}\eta\Omega + \left\{ \frac{3}{2\sqrt{h}}\delta\eta - C \right\} \left(n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} \right) \eta \\ + \frac{h}{6} \{h(\phi_x^2 + \phi_y^2)\}^{3/2} \eta_{sss} = 0 \end{aligned} \quad (3.9)$$

Multiplying this by 2η we write another relation

$$\frac{d\eta^2}{dT} - 2\sqrt{h}\eta\Omega + \left\{ \frac{3}{\sqrt{h}}\delta\eta^2 - \eta C \right\} \eta_s + \frac{h}{3} \{h(\phi_x^2 + \phi_y^2)\}^{3/2} \eta\eta_{sss} = 0 \quad (3.10)$$

Since C, n_1 and n_2 can be treated as constants in N_p , the operator $\frac{d}{dT}$ and integration with respect to s or $\zeta = \phi_0 - \beta\delta ct$ commute. Further, both integrations $\int_{-\infty}^{\infty} (\eta_0 - \eta_c\eta_0)\eta_0 ds$ and $\int_{-\infty}^{\infty} \eta_0\eta_0 s s s ds$ vanish. Hence, integrating (3.10) with respect to s from $-\infty$ to ∞ , we get

$$\frac{dD_2}{dT} - 2\sqrt{h}\Omega D_2 = 0 \quad (3.11)$$

where

$$D_2 = \int_{-\infty}^{\infty} \eta_0^2(s') ds' , \quad s' = s - \beta ct \quad (3.12)$$

(3.11) is a very important relation and implies that the product of D_2 and the ray tube area \mathcal{A} (which for the propagating curve Ω_t in (x, y) -plane may be taken to be g) associated with the crest-line is constant (see (2.2.23) in Prasad, 2001 and also Whitham, 1974). Using the expressions (3.4) we get

$$D_2 = 24c^{3/2}\alpha^2\beta \quad (3.13)$$

so that (3.6) and (3.11) give

$$\frac{dm}{dT} = \frac{4}{3}\sqrt{h}\Omega(m-1) \quad (3.14)$$

Now, we define a new time variable $\tau = t/\sqrt{h}$ and use the ray coordinate system (ξ, τ) in which the time rate of change is denoted by the partial derivative $\frac{\partial}{\partial \tau}$, so that $\frac{d}{dT} = \sqrt{h}\frac{\partial}{\partial \tau}$. The ray equations for the crest-line are

$$x_\tau = m \cos \theta , \quad y_\tau = m \sin \theta \quad (3.15)$$

From (1.1) we get the time rate of change of θ and g along the rays in the form

$$\theta_\tau = -\frac{1}{g}m\xi , \quad g_\tau = m\theta\xi \quad (3.16)$$

(see also equation (3.3.15, 16 and 19) Prasad, 2001). Since $\Omega = -\frac{1}{2g} \frac{\partial \theta}{\partial \xi}$, elimination of θ from the above equations gives

$$m_\tau = -\{2(m-1)/(3mg)\}g_\tau \quad (3.17)$$

which finally leads to the expression (1.2) for g . Note that the constant of integration in (3.17) can be chosen to be one by suitable choice of ξ .

We have now a complete formulation of the problem. Given initial position of the crest-line in terms of a parameter $\xi : x|_{\tau=0} = x_0(\xi)$, $y|_{\tau=0} = y_0(\xi)$ and amplitude distribution $m_0(\xi)$ on it, we determine initial value of g from (1.2). Then we first solve the system of conservation laws (1.1) and next we use the (3.15) to get the position of the crest-line at any time $t = \sqrt{h} \tau$ (for details, see chapter 6 in Prasad, 2001)

Since the local KdV has infinity of conservation laws, we can get as many transport equations as we wish, like (3.11). Infact, starting with equation (2.7), we shall get $g = (m-1)^{-1}e^{-(m-1)}$. But only one of these, namely (1.2) appears to be physically realistic. We first notice that $\delta\eta$ is proportional to the potential energy of the water per unit surface area of the water and $\frac{1}{2}\{(\delta\tilde{u})^2 + (\delta\tilde{v})^2\}$ is proportional to the corresponding kinetic energy. Therefore, to the first order in δ , the total energy density is $\delta\eta$ and the flux of the energy density crossing the lines parallel to the crest-line is proportional to

$$\{\delta(n_1\tilde{u} + n_2\tilde{v})\}\{\delta\eta\} + O(\delta^3) = (\delta^2/h)\eta^2 + O(\delta^3) \quad (3.18)$$

However, at the micro-scale of order ϵ_1 , $\eta^2\mathcal{A} = \eta^2g$ is not constant due to the presence of the dispersion term in (3.2). D_2 is proportional to the integral of the square of the flux at the micro-scale. Thus, (3.11) is the physically realistic transport equation along the rays associated with the crest-line. Result (1.2) is valid for $0 < m-1 \ll 1$, in which case we get $g(m) \simeq (m-1)^{-3/2}$. It is interesting to note that it agrees with the A - M relation used by Ostrovskii and Shrira (1976) and Miles (1977).

4. Geometrical shapes of the crest-line

Consider the parametric representation of the initial wavefront in the (x, y) -plane to be

$$x(\xi, 0) = \begin{cases} 0 & \text{if } \xi < 0 \\ -\xi g_r \sin \theta_r & \text{if } \xi > 0 \end{cases}$$

$$y(\xi, 0) = \begin{cases} g_l \xi & \text{if } \xi < 0 \\ \xi g_r \cos \theta_r & \text{if } \xi > 0 \end{cases}$$

which is equivalent to the initial data

$$(m, \theta)(\xi, 0) = \begin{cases} (m_l, 0) & \text{if } \xi < 0 \\ (m_r, \theta_r) & \text{if } \xi > 0 \end{cases} \quad (4.1)$$

for the system (1.1) in the (ξ, t) -plane, where m_l, m_r and θ_r are constants, $g_l = g(m_l)$ and $g_r = g(m_r)$ with g as defined in (1.2).

The initial value problem (1.1) together with the initial data (4.1) has been studied by Baskar and Prasad (2002) in the case of a general metric g satisfying a set of assumptions and shapes of the curve Ω_t for $t > 0$ has been computed for

$$g(m) = (m-1)^{-2}e^{-2(m-1)}$$

which appears for a front in weakly nonlinear ray theory in gas dynamics. As discussed in this work, the $(m > 1, \theta)$ -plane is divided into four regions A, B, C and D (as shown in Fig: 1) by four curves $R_i(U_i), S_i(U_i), i = 1, 2$, which are given by

$$\begin{aligned} R_1(U_1) &= \left\{ (m, \theta) \mid \begin{array}{l} 1 < m \leq m_l, \\ \theta = \sqrt{6(m_l - 1)} - \sqrt{6(m - 1)} \end{array} \right\} \\ R_2(U_1) &= \left\{ (m, \theta) \mid \begin{array}{l} m_l \leq m < \infty \\ \theta = \sqrt{6(m - 1)} - \sqrt{6(m_l - 1)} \end{array} \right\} \\ S_1(U_1) &= \left\{ (m, \theta) \mid \begin{array}{l} m_l \leq m < \infty \\ \theta = -\cos^{-1} \left(\frac{m_l g_l + m g(m)}{m g_l + m_l g(m)} \right) \end{array} \right\} \\ S_2(U_r) &= \left\{ (m, \theta) \mid \begin{array}{l} 1 < m \leq m_l \\ \theta = -\cos^{-1} \left(\frac{m_l g_l + m g(m)}{m g_l + m_l g(m)} \right) \end{array} \right\}, \end{aligned}$$

the line $\theta = -\pi$ and the curve

$$\theta = \begin{cases} \sqrt{6(m_l - 1)} + \sqrt{6(m - 1)}, & \text{for } \sqrt{6(m_l - 1)} < \theta < \pi \\ \pi, & \text{elsewhere} \end{cases}$$

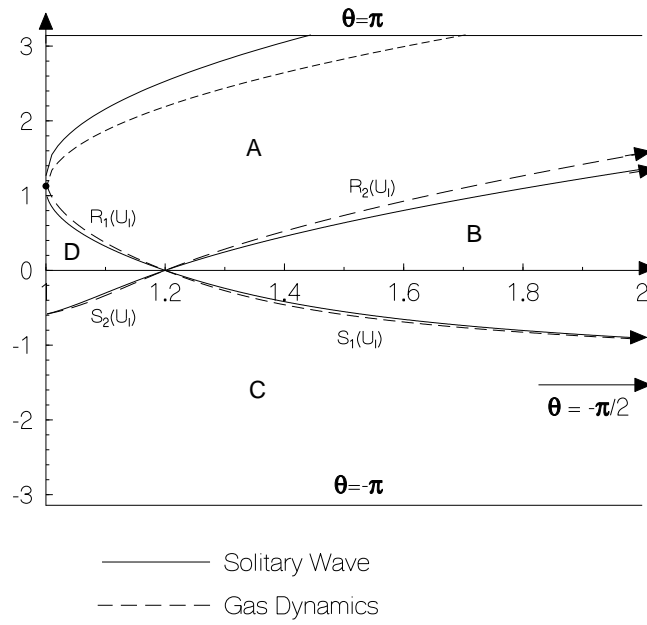
where we have taken the positive determination of \cos^{-1} .

The existence and uniqueness of the intermediate state through which the left state can be joined to the right state has been discussed by Baskar and Prasad (2001). Their results with general function g show that qualitatively the shapes of Ω_t with *similar* initial conditions are the same. Since the curves R_1, R_2, S_1 and S_2 differ by a small quantity for two expressions for g , we do not expect much quantitative change also.

If the right state (m_r, θ_r) lies on $R_i(U_i), i = 1, 2$, then the singularity on the initial wavefront will be resolved and the wavefront becomes smooth for $t > 0$. This smooth part denoted as elementary shape \mathcal{R}_i ($i = 1, 2$), on the wavefront is the image of the rarefaction region in the (ξ, t) -plane. These are depicted in Fig: 2 along with the results for the nonlinear wavefront in gasdynamics.

If the right state (m_r, θ_r) lies on $S_i(U_i), i = 1, 2$ then the singularity on Ω_0 remains unresolved and propagates on Ω_t as a \mathcal{K}_i ($i = 1, 2$), elementary shape which is a kink. A comparison between the crest-line of the solitary wave and nonlinear wavefront in gas dynamics is shown in the figure 3.

In all the above cases the shape of a propagating crest-line is qualitatively same as that of a nonlinear wavefront in the gas dynamics. Infact, had we taken the initial position and initial shapes to be the same (which can be easily done) we would not have found much difference in the graphs. Therefore we omit the detailed discussion of different shapes except showing their graphs in Fig. 4. of the Ω_t in the present case when the right state (m_r, θ_r) lies in the regions A, B, C and D . For the full details, we refer to Baskar and Prasad, 2002.



[h]

FIG 1: Domain A, B, C, D in (m, θ) -plane with $m_l = 1.2$ and $\theta_l = 0$. When (m_r, θ_r) belong to these domains, we get different geometrical shapes.

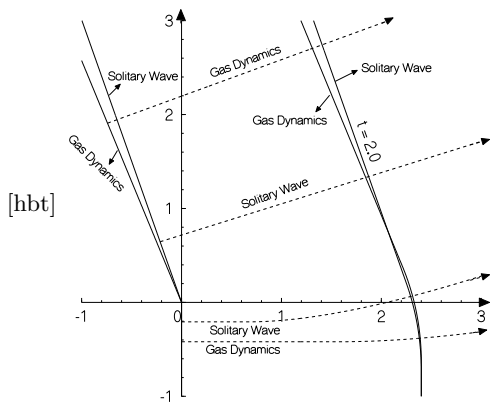


FIG 2a. Comparison of \mathcal{R}_1 elementary shape

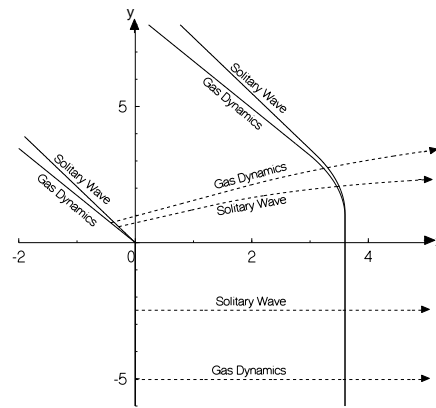


FIG 2b. Comparison of \mathcal{R}_2 elementary shape

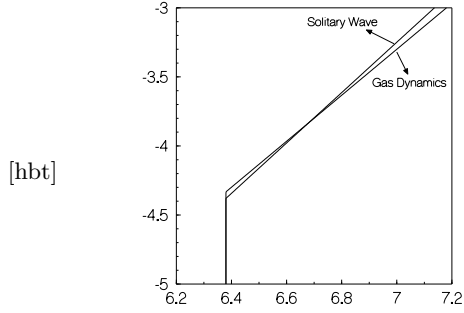


FIG 3a. Comparison of \mathcal{K}_1 elementary shape

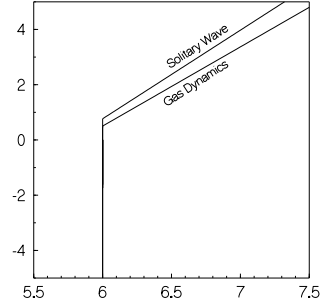


FIG 3b. Comparison of \mathcal{K}_2 elementary shape

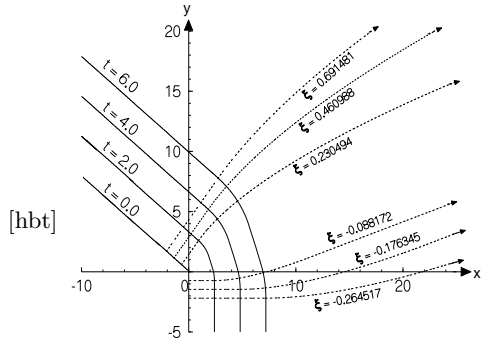


FIG 4a. Propagation of the crest-line when $(m_r, \theta_r) \in A$ with $m_l = 1.2$, $m_r = 1.3$, $\theta_r = 0.9$.

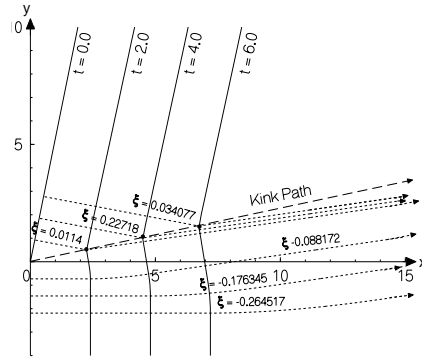


FIG 4b. Propagation of the crest-line when $(m_r, \theta_r) \in B$ with $m_l = 1.2$, $m_r = 1.7$, $\theta_r = 0.5$.

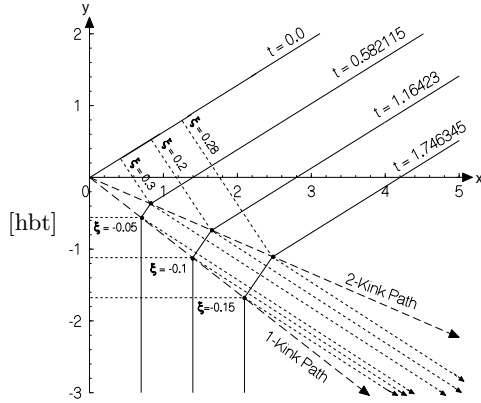


FIG 4c. Propagation of the crest-line when $(m_r, \theta_r) \in C$ with $m_l = 1.2$, $m_r = 1.3$, $\theta_r = -0.9$.

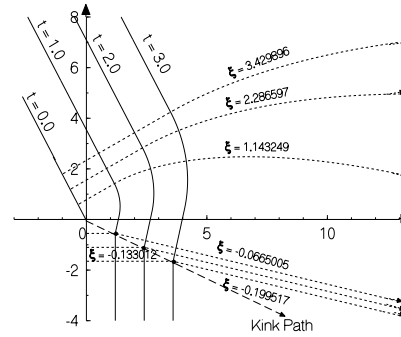


FIG 4d. Propagation of the crest-line when $(m_r, \theta_r) \in D$ with $m_l = 1.2$, $m_r = 1.05$, $\theta_r = -0.2$.

Acknowledgement: We Sincerely thank Professors D. H. Peregrine and V. I. Shrira for bringing to our notice the important developments on the subject in water waves. We also thank Prof. J. L. Bona for valuable discussion.

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