A shock ray theory for Propagation of a curved shock of arbitrary strength

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1 Introduction

Calculating successive positions of a curved shock has been one of the most difficult problems in the theory of nonlinear hyperbolic waves. Since the problem has many practical applications, in which knowledge of finer geometrical features of a shock front is required (Apazidis et al (2002), Higashino (2001), Plotkin (2002), Schwendeman (2002)), many computational attempts have been made to solve this problem. A numerical method starting from the original conservation laws (e.g. compressible Euler's equations) of the physical system in which the shock front appears, is not only computationally time consuming but also does not give sufficiently sharp shock front which can be used to determine finer geometrical features of the front. Since a shock front is a surface of discontinuity, the high frequency approximation is exactly satisfied (not just approximately) and a shock ray theory (SRT) appears to be ideally suited for this problem. The exact SRT consists of the ray equations (Prasad, 1982) and an infinite system of compatibility conditions along the shock rays (Grienfel'd (1978), Maslov (1960)). Unfortunately, the compatibility conditions are too complex to be evaluated beyond the second compatibility conditions (See Prasad (2001) for details). It is also seen in a preliminary investigation that the infinite system of equations are not well posed unless we seek a solution in the class of analytic (regular) functions (Ravindran and Prasad (1990)). This class of functions is too restrictive from the point of view of applications. However, as discussed by Ravindran and Prasad (1990)- see section 7.2 of Prasad (2001), a truncated system of the compatibility conditions may lead to an approximate theory. This is indeed true, an approximate SRT for a weak shock with just two compatibility conditions is extremely useful (Monica and Prasad (2001)) and gives results which agree well with experimental results, known exact results and with results obtained by numerical solution of Euler's equations (Kevlahan (1996) and Baskar and Prasad (2003)). This success prompts us to develop a SRT for a curved shock of arbitrary strength.

The compatibility conditions along shock rays for a weak shock follow a simple pattern and can be easily derived (Monica and Prasad (2001)). Infinite system of compatibility conditions for a weak shock showing this simple pattern was first derived by Anile and Russo (1986) but they wrote them along the linear rays. Such a simple pattern is not available for a shock of an arbitrary strength (Ravindran and Prasad (1993), Lazarev, Ravindran and Prasad (1998); Chapter 9 of Prasad (2001)). We take here the first two compatibility conditions and show that we can obtain an approximate and not too complicated - SRT under the assumptions that

(i) the shock front consists of a number of segments, two segments, if they meet, they meet at a kink, and

(ii) each segment has a small curvature and the variation of the shock strength is small on it.

These assumptions do not rule out that when all sections are taken together, the shock front may be highly curved or even form a closed cylinder (we shall consider only a two-dimensional shock fronts). Both assumptions are justified from a large number of computational and experimental results (Apazidis etal(2002), Monica and Prasad (2001)). Every shock front, however smooth but which has a non-uniform distribution of the shock strength on it or which is not perfectly circular is expected to ultimately evolve into curves which satisfy both (i) and (ii) due to its corrugational stability.

When the gradient of the solution behind a shock moving into a uniform state is large and the shock strength is not small, the strength either decays quickly to a small value or blows up. We shall like to study the geometrical shape and the shock strength distribution on it during the period of rapid decay of the shock strength. Therefore, we make one more assumption:

(iii) the state behind the shock satisfies the high frequency or short wave assumption so that the time scale involved in the procedure is small.

The SRT (with two compatibility conditions) for a shock of arbitrary strength which we have formulated appears to be ideally suited to study this problem. Once the shock strength becomes quite weak, the weak SRT applies and further evolution no longer takes place in a short time.

As mentioned earlier, derivation of the compatibility conditions along shock rays is very tedious due to appearance of long and complex expressions of the coefficients. We use the explicit form of the first two compatibility conditions derived by Ravindran and Prasad (1993). However, we could have also used the form of the compatibility conditions derived by Lazarev (see chapter 5 in Prasad (1993) or chapter 9 in Prasad (2001)) or in a more general setting Lazarev, Ravindran and Prasad(1998). The later formulation has been verified to give very good results for a shock of arbitrary strength for a plane shock (Lazarev, Prasad and Singh (1995); see chapter 8, Prasad (2001)). We expect the 2-D generalization of this formulation to be even more successful.

2 Basic equations of the SRT with two compatibility conditions

Consider the propagation of a cylindrical shock front denoted Ω_t in a polytropic gas such that the state ahead of the shock is uniform and in equilibrium with $(\rho, \mathbf{q}, p) = (\rho_0, \mathbf{0}, p_0)$ where ρ is the density, $\mathbf{q}$ the fluid velocity and p the pressure. The motion of such a shock front can be described in $(\mathbf{x}, t)$-space, where $\mathbf{x} = (x, y)$ are spatial coordinates. Let $\mathbf{N} = (N_1, N_2)$ be the unit normal to the shock front directed into the gas in front of the shock. Let ρ_1 be the density behind the shock. We define two variables, μ and $\bar{\mu_1}$ on the shock front Ω_t

$$\mu = \frac{\rho_l - \rho_0}{\rho_0}|_{\Omega_t - 0} \ , \ \bar{\mu_1} = \frac{1}{\rho_0}\left(N_1\frac{\partial \rho}{\partial x} + N_2\frac{\partial \rho}{\partial y}\right)\Bigg|_{\Omega_t - 0} \tag{2.1}$$

where $|_{\Omega_t - 0}$ on a quantity means the limiting value of the quantity as we approach the shock front from behind the shock. Let L be a length scale in the problem. Then we non-dimensionalize spatial coordinates by L and time by L/a_0 where a_0 is the sound velocity in the uniform state i.e., $a_0^2 = \frac{\gamma p_0}{\rho_0}$ with γ as the ratio of specific heats and denote these non-dimensional variables by the same symbols x, y and t. We also introduce a non-dimensional μ_1 by

$$\mu_1 = L\bar{\mu_1} \tag{2.2}$$

and use only non-dimensional variables from now onwards.

The shock Mach number, denoted by M, is the ratio of the shock speed C to a_0 and hence is given by

$$M = \{2(1+\mu)/Q\}^{1/2} \ , \ Q = 2 - (\gamma - 1)\mu \tag{2.3}$$

For a derivation of this and all results in this section, see Ravindran and Prasad (1993). We introduce now a symbol

$$S = 8 + 5\mu - 3\mu\gamma + \mu^2(\gamma^2 - 1) \tag{2.4}$$

Let Θ be the angle which a shock ray (i.e. normal to Ω_t for the state at rest ahead of the shock) makes with the x-axis and (X, Y) be a point Ω_t. Then the shock rays are given by

$$\frac{dX}{dt} = M\cos\Theta \ , \ \frac{dY}{dt} = M\sin\Theta \tag{2.5}$$

and

$$\frac{d\Theta}{dt} = -\frac{(\gamma+1)M}{2(1+\mu)Q}\frac{\partial \mu}{\partial T} = -\frac{\partial M}{\partial T} \tag{2.6}$$

where the operator $\frac{\partial}{\partial T}$, representing tangential derivative on Ω_t is given by

$$\frac{\partial}{\partial T} = \cos\Theta\frac{\partial}{\partial y} - \sin\Theta\frac{\partial}{\partial x} \tag{2.7}$$

The first two compatibility conditions, after omitting the second derivative term

$$\mu_2 = \left\{ \left(N_1^2 \frac{\partial^2}{\partial x^2} + 2N_1N_2 \frac{\partial^2}{\partial x \partial y} + N_2^2 \frac{\partial^2}{\partial y^2} \right) \left(\frac{\rho - \rho_0}{\rho_0} \right) \right\} \Big|_{\Omega_t - 0} \tag{2.8}$$

in the second compatibility condition, are

$$\frac{d\mu}{dt} = -\frac{\mu Q M}{S} \left(2\frac{\partial \Theta}{\partial T} + \frac{\gamma + 1}{1 + \mu} \mu_1 \right) \tag{2.9}$$

and

$$\begin{aligned} \frac{d\mu_1}{dt} = & -\frac{M}{\zeta_1} \left\{ -\zeta_3 \mu_1 \frac{\partial \Theta}{\partial T} + \zeta_2 \mu_1^2 \right. \\ & \left. + \zeta_4 \left(\frac{\partial \Theta}{\partial T} \right)^2 + \zeta_5 \frac{\partial^2 \mu}{\partial T^2} + \zeta_6 \left(\frac{\partial \mu}{\partial T} \right)^2 \right\} \end{aligned} \tag{2.10}$$

where the coefficients $\zeta_1, \zeta_2, \ldots, \zeta_6$ are functions of μ and are given in the Appendix.

The basic assumptions on which we shall like to develop the SRT for a shock of arbitrary strength have already been stated in the introduction as assumptions (i), (ii) and (iii). Assumptions (i) and (ii) require special consideration and are to be considered together. Since different, almost plane, sections of the shock front meet at kinks making not necessarily small angles, we can not neglect all terms in (2.9) and (2.10) on the assumption that $\frac{\partial \Theta}{\partial T}$ (and also other quantities such as $\frac{\partial^2 \mu}{\partial T^2}$) are very small. If we do so we shall completely miss the kink phenomenon. Hence, we use the assumption (i) and (ii) only to neglect some terms. More specifically, we take

$$\left\{ \left(\frac{\partial \Theta}{\partial T} \right)^2, \left(\frac{\partial \mu}{\partial T} \right)^2, \frac{\partial^2 \mu}{\partial T^2} \right\} = o \left(\frac{\partial \Theta}{\partial T} \right) \tag{2.11}$$

so that all the three quantities on the right hand side can be neglected in comparison with $\frac{\partial \Theta}{\partial T}$. We expect that in the *structure of a kink*, which would imply replacing the kink by a curve with continuously turning tangent, $\frac{\partial \Theta}{\partial T}$ would not be small but the quantities on the left hand side of (2.11) would also not be small and comparable to $\frac{\partial \Theta}{\partial T}$ so that they will have to be retained to get the structure of the kink. Therefore, the assumption (ii) is meant to simplify the computation in that part of the shock front where the terms on the left hand side of (2.11) are negligible and to ignore them once more near a kink since we are no longer interested in the *kink structure*. On these intuitive arguments, whose validity has to be seen from the results we get from this theory, we approximate the equation (2.10) by

$$\frac{d\mu_1}{dt} = \frac{M\zeta_3}{\zeta_1} \mu_1 \frac{\partial \Theta}{\partial T} - \frac{M\zeta_2}{\zeta_1} \mu_1^2 \tag{2.12}$$

We discuss now the assumption (iii) mentioned in the introduction. For a shock of finite amplitude we usually have $\mu_1 = O(1)$. But the high frequency assumption implies

$\mu_1 = O\left(\frac{1}{\epsilon}\right)$, where ϵ is the short length scale of variation of the solution behind the shock. Since μ_1 is large, equation (2.9) shows that μ will decay to a small quantity over a small time of order ϵ. Hence, we introduce a new time t' and new μ_1

$$t' = t/\epsilon \quad , \quad \mu_1' = \epsilon\mu_1 \tag{2.13}$$

In a smooth portion of the shock front on either side of a kink, the first term in the bracket on the right hand side of (2.9) is negligible. But at the kink, Θ is discontinuous and hence to describe the evolution of a shock front with kinks, we introduce the scaling (2.13) in (2.9) and (2.12) but we do not neglect the terms containing $\frac{\partial\Theta}{\partial T}$ which we assume to be O(ϵ) away from a kink. Thus we assume that in the rapid decay of the shock strength on a curved shock, all terms in (2.9) and (2.12) are equally important. Over a short time scale of the order of ϵ, the changes in X, Y and Θ at a point of the shock front will, also be small (i.e., the displacement of the front in the normal direction will be small). So, we are interested in propagation of the shock front over a small distance along a ray. Now we note that the changes in X, Y and Θ with respect to t' be of the order of unity and the left hand side terms in equations in (2.5) and (2.6), balance those on the right hand side.

We first transform these equations in new variables t' and μ_1' and since the equations remain the same we drop 'dash' from these variables. The final equations giving the rapid decay of a curved shock with high frequency waves behind it, are (2.5), (2.6), (2.9) and (2.12). In these equations, all quantities including $\frac{\partial\theta}{\partial T}$ are now treated to be of order 1.

3 Ray coordinate system and conservation form of the equations

We introduce the ray coordinate system (ξ, t) in usual way (Prasad, 2001). Here $t =$ constant give successive positions of the shock front in (x, y)-plane and $\xi =$ constant give rays, in this case the rays are orthogonal to the shock positions. Let G denote the metric associated with the coordinate ξ, so that $\frac{\partial}{\partial T} = (1/G)\frac{\partial}{\partial\xi}$ then from (3.3.15) with $T = 0$ of Prasad (2001), we get

$$G_t = M\Theta_\xi \tag{3.1}$$

Note that the time derivative $\frac{d}{dt}$ along the ray is a partial derivative in the ray coordinate system (2.6) and the two compatibility conditions (2.9) and (2.12) become

$$\Theta_t + \frac{(\gamma+1)M}{2(1+\mu)QG}\mu_\xi = 0 \tag{3.2}$$

$$\mu_t + \frac{2\mu QM}{SG}\Theta_\xi = -\frac{(\gamma+1)M\mu Q}{(1+\mu)S}\mu_1 \tag{3.3}$$

and

$$\mu_{1t} - \frac{\mu_1 \zeta_3 M}{\zeta_1 G}\Theta_\xi = -\frac{M\zeta_2}{\zeta_1}\mu_1^2 \tag{3.4}$$

The system of four equations (3.1) - (3.4) is hyperbolic with eigenvalues

$$c_1 = -\left\{\frac{(\gamma+1)\mu M^2}{(1+\mu)SG^2}\right\}^{1/2}, \; c_2 = \left\{\frac{(\gamma+1)\mu M^2}{(1+\mu)SG^2}\right\}^{1/2}, \; c_3 = 0, c_4 = 0 \tag{3.5}$$

where we note that $\mu > 0$ and $S > 0$ so that c_1 and c_2 are real. For a weak shock $\mu = O(M-1) << 1$ and $c_1 \sim \{(M-1)/2G^2\}^{1/2}$ which is same as the eigenvalue for the weak SRT (see Monica and Prasad, 2001).

Using (3.1), we eliminate Θ_ξ from (3.3) and (3.4) to get

$$\frac{S}{2\mu Q}\mu_t + \frac{1}{G}G_t = -\frac{(\gamma+1)M}{2(1+\mu)}\mu_1 \tag{3.6}$$

and

$$\mu_{1t} - \frac{\mu_1\zeta_3}{\zeta_1 G}G_t = -\frac{M\zeta_2}{\zeta_1}\mu_1^2 \tag{3.7}$$

We rearrange terms in the equation (3.6) in a suitable form

$$\frac{2}{\mu}\mu_t + \frac{G_t}{G} + \{F_0'(\mu)/F_0(\mu)\}\mu_t = -\frac{(\gamma+1)M}{2(1+\mu)}\mu_1 \tag{3.8}$$

where

$$\frac{F_0'(\mu)}{F_0(\mu)} = \frac{S}{2\mu Q} - \frac{2}{\mu} = \frac{S-4Q}{2\mu Q} \Rightarrow F_0(\mu) = \frac{(\gamma+1)^2}{16}\exp\left\{\int_0^\mu \frac{S-4Q}{2\mu Q}d\mu\right\} \tag{3.9}$$

with a proper choice of the constant of integration for $F_0(\mu)$.

For future reference, we rewrite (3.7) also in a suitable form. We first write it as

$$\mu_{1t} + \frac{\mu_1}{2G}G_t = \left(\frac{\mu_1\zeta_3}{\zeta_1 G} + \frac{\mu_1}{2G}\right)G_t - \frac{M\zeta_2}{\zeta_1}\mu_1^2$$

and then eliminate G_t/G on the right hand side of it by use of (3.6). This gives

$$\begin{aligned}\frac{2\mu_{1t}}{\mu_1} + \frac{G_t}{G} \;&+\; \frac{F_1'(\mu)}{F_1(\mu)}\mu_t \\ =\; &-\; \left\{\frac{(\gamma+1)(2\zeta_3+\zeta_1) + 4(1+\mu)\zeta_2}{2(1+\mu)}\right\}\frac{M}{\zeta_1}\mu_1\end{aligned} \tag{3.10}$$

where

$$F_1'(\mu)/F_1(\mu) = \frac{S(2\zeta_3+\zeta_1)}{2\mu\zeta_1 Q} \Rightarrow F_1(\mu) = \exp\left\{\int_0^\mu \frac{(2\zeta_3+\zeta_1)S}{2\mu\zeta_1 Q}d\mu\right\}. \tag{3.11}$$

We have now derived suitable forms of equations to deduce their conservation forms, which we think are physically realistic. First we note that a two-dimensional shock front is a particular case of a moving curve for which there exists a pair of kinematical conservation laws in a ray coordinate system. These conservation laws are physically realistic as they conserve distances in two independent directions in a special sense.

Taking the shock front as a particular case of an arbitrary moving curve, the kinematical conservation laws (KCL) from which the differential equations (3.1) and (3.2), where the the second term on the left hand side of (3.2) is equal to $-(M_\xi/G)$, can be derived are

$$(G\sin\Theta)_t + (M\cos\Theta)_\xi = 0 \tag{3.12}$$

$$(G\cos\Theta)_t - (M\sin\Theta)_\xi = 0 \tag{3.13}$$

We shall now derive two more conservation forms which are equivalent in differential form to the partial differential equations (2.9) or (3.6) and (2.12) or (3.7).

We notice that the second term on the left hand side of (3.8) represents a geometric amplification or decay of a quantity which will be decided by the first and the third terms. From (3.8) we deduce the following conservation form

$$(\mu^2 G F_0(\mu))_t = -F_2(\mu)\mu_1 \tag{3.14}$$

where

$$F_2(\mu) = -\frac{(\gamma+1)\mu^2 G F_0(\mu) M(\mu)}{2(1+\mu)} \tag{3.15}$$

Similarly, we derive from (3.10)

$$(\mu_1^2 G F_1(\mu))_t = -F_3(\mu)\mu_1^3 \tag{3.16}$$

where $F_3(\mu)$

$$F_3(\mu) = \frac{\{(\gamma+1)(2\zeta_3+\zeta_1) + 4(1+\mu)\zeta_2\} G M(\mu) F_1(\mu)}{2(1+\mu)\zeta_1} \tag{3.17}$$

This completes the formulation of a second order SRT in conservation form. We refer to it as second order SRT because we have used only two of the infinite system of equations after truncating the coupling term in the second compatibility condition. Given a suitable problem for the Euler's equations with initial and boundary conditions, we should be able to derive the initial values of μ (or $M(\mu)$) and μ_1 in the ray coordinates (ξ, t) and solve the four conservation laws (3.12) - (3.17). The successive positions of the shock and the shock rays and hence μ and μ_1 on the shock can be obtained by integrating the equations (2.5).

4 Simplification of the functions appearing in the equations

The expressions for the coefficients $\zeta_1, \zeta_2, \ldots, \zeta_6$ in the second compatibility condition (2.10) are too long and complex to be of any use in theory and they will also require considerable time in numerical computation. In this section we examine the behavior of these function in the range $0 < \mu < 5$ for $\gamma = 1.4$. We may use interpolation formulae to replace them by suitable polynomials in μ which approximate them over this range of μ or another sub-interval of (0,5) which will be of interest in a particular problem. The functions, which we need to simplify by low degree polynomial expressions in μ are $\zeta_1, \zeta_2, \zeta_3; F_0, F_1, F_2, F_3$. For the kink structure, we also need approximate low degree polynomial expressions for ζ_4, ζ_5 and ζ_6.

To draw graphs of the above functions with μ

5 The jump conditions

Let the subscripts l and r represent values on two sides of a shock. Following the results from Prasad (2001), we find the following jump relations from KCL

$$\cos(\Theta_r - \Theta_l) = \frac{M_l G_l + M_r G_r}{M_l G_r + M_r G_l} \tag{5.1}$$

$$s^2 = \frac{M_l^2 - M_r^2}{G_r^2 - G_l^2} \tag{5.2}$$

and

$$s = \frac{M_r^2 - M_l^2}{(M_l G_r + M_r G_l)\sin(\Theta_r - \Theta_l)} \tag{5.3}$$

where s is the shock velocity in (ξ, t)-plane i.e., in terms of the shock position $\xi_s(t), s = d\xi_s/dt$. For $s \neq 0$, the jump relations from (3.14) and (3.16) give

$$\mu_l^2 G_l F_0(\mu_l) = \mu_r^2 G_r F_0(\mu_r) \tag{5.4}$$

and

$$\mu_{1l}^2 G_l F_1(\mu_l) = \mu_{1r}^2 G_r F_1(\mu_r) \tag{5.5}$$

Eliminating the ratio G_r/G_l between (5.1) and (5.4), we get the Hugoniot curve: for a fixed μ_l and Θ_l, we get Θ_r as a function of μ_r. Similarly, eliminating G_r/G_l between (5.4) and (5.5), we get

$$\mu_{1r}^2 = \mu_{1l}^2 \left(\frac{\mu_r}{\mu_l}\right)^2 \frac{F_0(\mu_r)}{F_0(\mu_l)} \frac{F_1(\mu_l)}{F_1(\mu_r)} \tag{5.6}$$

so that we explicitly get μ_{1r} as a function of μ_l, μ_r and μ_{1l}. It is interesting to note that the Hugoniot relation between Θ_r and μ_r is independent of μ_{1l} and μ_{1r}

To Draw
(i) graphs of Θ_r with μ_r for some values of μ_l and $\Theta_l = 0$
(ii) graphs of μ_{lr} with μ_r for some values of μ_l, μ_{1l} and $\Theta_l = 0$.

6 Shock ray theory for a weak shock

We first notice that for a weak shock i.e., $0 < \mu \ll 1$, the relation between M and μ in (2.3) up to first order terms becomes

$$M = 1 + \frac{\gamma + 1}{4}\mu \tag{6.1}$$

We also note that in Baskar and Prasad (2004), we used a quantity V, where

$$V = \frac{\gamma + 1}{4}\mu_1 \tag{6.2}$$

In the linear theory, consideration of propagation of energy along a ray tube leads to the relation $(M-1)^2G$=constant i.e., $\{(M-1)^2G\}_t = 0$. Nonlinearity (or more precisely the genuine nonlinearity in the characteristic field) modifies this result in two ways. Firstly, it gives a term $-F_2(\mu)\mu_1$ on the right hand side of (3.14). This term represents dissipation of the total (internal and kinetic) energy through the shock. Secondly, it modifies a properly scaled total energy from its value $(M-1)^2 = \frac{(\gamma+1)^2}{16}\mu^2$ in the linear theory to a new value E_s given by

$$E_s = \mu^2 F_0(\mu) \tag{6.3}$$

If we retain the most dominant term in (6.2) for $0 < \mu \ll 1$, we get $E_s \sim \frac{(\gamma+1)^2}{16}\mu^2$ which is the linear value. But, to the next order, (6.2) gives a value E_{s1}, where

$$E_{s1} = \frac{(\gamma + 1)^2}{16}\mu^2 \exp\left(\frac{\gamma + 1}{4}\mu\right) \tag{6.4}$$

In order to compare the present results to our previous ones, we write an approximate form of (3.14) in terms of M and V:

$$\{G(M-1)^2 e^{(M-1)}\}_t + 2GM(M-1)^2 e^{(M-1)}V = 0 \tag{6.5}$$

The corresponding result of Baskar and Prasad (2004) for a weak shock ray theory is

$$\{G(M-1)^2 e^{2(M-1)}\}_t + 2GM(M-1)^2 e^{2(M-1)}V = 0 \tag{6.6}$$

The two equations (6.5) and (6.6) agree up to leading terms when we note that for a weak shock $0 < M - 1 \ll 1$, so that $e^{2(M-1)} \simeq 1 \simeq e^{(M-1)}$. Two different equations

(6.5) and (6.6) would not appear if the basic equation (2.6) of Baskar and Prasad (2004), namely

$$M_t + \frac{1}{2G}(M-1)\Theta_\zeta + (M-1)V = 0 \tag{6.7}$$

which gives the equation (6.6), is replaced by

$$\frac{1}{2}(M+1)M_t + \frac{M}{2G}(M-1)\Theta_\zeta + M(M-1)V = 0 \tag{6.8}$$

which gives (6.5).

Our previous theory was developed assuming small amplitude i.e., $M-1 = 0(\epsilon) \ll 1$ from the very beginning. In this case, the coefficient of M_t in (6.8) agrees with that in equation (6.7) up to first order terms. We find the same agreement between (6.5) and (6.6) as $e^{(M-1)}$ and $e^{2(M-1)}$ both have value $1 + 0(\epsilon)$. In our previous numerical work (Prasad and Sangeeta (1999), Monica and Prasad (2001) and Baskar and Prasad (2004)), we did not approximate $G(M-1)^2 e^{2(M-1)}$ by $G(M-1)^2 + O(\epsilon^3)$ because the numerical methods for the conservation forms were based on the use of the characteristic fields of the original simple and elegant hyperbolic system of equations.

Similarly we can discuss the limiting of the equation (3.16) for $0 < M - 1 \ll 1$ and compare the result with the equation (2.10) of baskar and prasad (2004). We have not checked the results but we expect the result to agree up to most dominant terms.

7 Kink structure

Without any loss of any generality, we can choose $\Theta_l = 0$.

Given two states $(\mu_l, \Theta_l = 0, \mu_{1l})$ and (μ_r, Θ_r, μ_1) satisfying the jump relations in the section 5, we can calculate s from (5.2). Then we look for a solution of equations (2.6), (2.9), (2.12) and (3.1) in a *steady* frame, in which all quantities are functions of $z = \xi - st$ joining the state $(\mu_l, 0, G_l, \mu_{1l})$ at $z = -\infty$ and $(\mu_r, \Theta_r, G_r, \mu_{1r})$ at $z = \infty$. We first note that

$$\begin{aligned} \frac{\partial^2 \mu}{\partial T^2} &= \left(-\frac{1}{G^3} G_\xi \frac{\partial}{\partial \xi} + \frac{1}{G^2}\frac{\partial^2}{\partial \xi^2} \right)\mu \\ &= \left(-\frac{1}{G^3} \frac{dG}{dz}\frac{d}{dz} + \frac{1}{G^2}\frac{d^2}{dz^2} \right)\mu \end{aligned} \tag{7.1}$$

The ordinary differential equations governing the shock structure are

$$\frac{1}{G}\frac{dM}{dz} - s\frac{d\Theta}{dz} = 0 \quad , \; s\frac{dG}{dz} + M\frac{d\Theta}{dz} = 0 \tag{7.2}$$

$$s\frac{d\mu}{dz} - 2\frac{\mu Q M}{SG}\frac{d\Theta}{dz} = \frac{(\gamma+1)QM\mu}{(1+\mu)S}\mu_1 \tag{7.3}$$

$$\frac{M}{\zeta_1}\left[\zeta_5\left(\frac{1}{G^2}\frac{d^2\mu}{dz^2}-\frac{1}{G^3}\frac{dG}{dz}\frac{d\mu}{dz}\right)+\frac{\zeta_4}{G^2}\left(\frac{d\Theta}{dz}\right)^2+\frac{\zeta_6}{G^2}\left(\frac{d\mu}{dz}\right)^2\right]$$
$$-\frac{M\zeta_3\mu_1}{\zeta_1}\frac{d\Theta}{dz}-s\frac{d\mu_1}{dz}=-\frac{M\zeta_2}{\zeta_1}\mu_1^2 \tag{7.4}$$

Eliminating $\frac{d\Theta}{dz}$ from (6.2) and integrating once, we get

$$s^2G^3+\frac{3}{2}M^2=s^2G_l^3+\frac{3}{2}M_l^2 \tag{7.5}$$

We can use this relation to solve G in terms of M and substitute in (7.3) and (7.4). We can also use the first equation in (7.2) to eliminate $\frac{d\Theta}{dz}$ from (7.3) and (7.4). Thus we get a first order equation (7.3) only in μ (or M) and then an equation (7.4) containing derivatives upto second order in μ and first order in μ_1. This *third order* system should satisfy four boundary conditions ($\mu=\mu_l, \mu_1=\mu_{1l}$) at $z=-\infty$ and ($\mu=\mu_r, \mu_1=\mu_{1r}$) at $z=\infty$. Thus we, should be able to solve the kink structure problem uniquely.

Acknowledgement: We thank AR&DB, Ministry of Defence, Govt. of India for financial support through the project "Nonlinear Hyperbolic Waves in Multi-Dimensions with Special Reference to Sonic Booms"(No. DARO/08/1031199/M/I).

Appendix
(from Ravindran and Prasad (1993))

$R = 4 + 3\mu - \mu\gamma + \mu^2(\gamma^2 - 1)$

$T = 4 + 6\mu - 2\mu\gamma + 3\mu^2(\gamma^2 - 1)$

$U = 5 - 3\gamma + 2\mu(\gamma^2 - 1)$

$$\begin{aligned}
\beta_1 &= \frac{1}{2(1+\mu)^2}\left[2 + \mu(\gamma+1) - \frac{\mu^3(\gamma+1)(\gamma^2-1)}{S}\right] \\
\beta_2 &= \frac{\mu(\gamma+1)}{(1+\mu)^2}\frac{Q}{S}\left\{\beta_1\left(2 - \frac{\gamma+1}{Q}\right) + \frac{\gamma+1}{2(1+\mu)S}\left(3\mu^2(\gamma^2-1) - S - \frac{\mu^3(\gamma^2-1)U}{S}\right)\right\} \\
\beta_3 &= \frac{\mu Q}{S(1+\mu)}\left\{\beta_1\left(-4 + 2\frac{\gamma+1}{Q}\right) - \frac{\gamma+1}{(1+\mu)S}\left(6\mu^2(\gamma^2-1) - S - 2\mu^3\frac{(\gamma^2-1)U}{S}\right.\right. \\
&\quad \left.\left. + \frac{(\gamma^2-1)(\gamma-1)\mu^3}{Q}\right)\right\} \\
\beta_4 &= \frac{\mu^3(\gamma^2-1)}{(1+\mu)S^2}\left\{S + 6Q + 2\mu(\gamma-1) - \frac{2\mu U Q}{S}\right\} \\
\beta_5 &= \frac{\mu^3(\gamma^2-1)(\gamma+1)}{2(1+\mu)^2 QS} \\
\beta_6 &= -\frac{\mu^3(\gamma^2-1)(\gamma+1)\{5 - 3\gamma - 4\mu(\gamma-1)\}}{4(1+\mu)^3Q^2S}
\end{aligned}$$

and

$$\begin{aligned}
\delta_1 &= \frac{1}{2(1+\mu)^2}\left\{2 + \mu(\gamma+1) - \frac{\mu(\gamma+1)R}{S}\right\} \\
\delta_2 &= \frac{\mu(\gamma+1)}{(1+\mu)^2}\frac{Q}{S}\left\{\delta_1\left(2 - \frac{\gamma+1}{2Q}\right) + \frac{\gamma+1}{2(1+\mu)S}\left(T - S - \frac{\mu R U}{S}\right)\right\} \\
\delta_3 &= \frac{\mu Q}{(1+\mu)S}\left\{\frac{\gamma+1}{\mu+1} - 4\delta_1 + \frac{\gamma+1}{2Q}\left(2\delta_1 - \frac{\gamma+1}{(\mu+1)^2}\frac{\mu R}{S}\right) + \frac{\mu(\gamma+1)R}{(1+\mu)^2S}\right) \\
&\quad \left. -2\frac{(\gamma+1)}{(1+\mu)S}\left(T - \frac{\mu R U}{S}\right)\right\} \\
\delta_4 &= \frac{\mu}{(1+\mu)S}\left\{R + \frac{2Q}{S}\left(T - \frac{\mu R U}{S}\right) + \frac{2\mu Q R}{(1+\mu)S}\left(\frac{\gamma+1}{2Q} - 1\right)\right\} \\
\delta_5 &= \frac{(1+\gamma)\mu R}{2(1+\mu)^2 QS} \\
\delta_6 &= -\frac{(\gamma+1)\{5 - 3\gamma - 4\mu(\gamma-1)\}\mu R}{4(1+\mu)^3Q^2S}
\end{aligned}$$

$$\zeta_1 = \delta_1 + \beta_1 - \frac{\mu(1+\gamma)}{2(1+\mu)^2}$$

$$
\begin{aligned}
\zeta_2 &= \delta_2+\beta_2+\frac{(\gamma+1)}{4(1+\mu)^4}\left\{(2+\mu)(2+\mu(1+\gamma))\right.\\
&+\frac{\mu}{S}[8(1+\mu)(\gamma-1)-2(4+3\mu-\mu\gamma)(4+\mu+\mu\gamma)\\
&-\mu^2(\gamma^2-1)(6+4\gamma+3\mu(\gamma+1))]+\frac{\mu^2(\gamma+1)}{S^2}\\
&\left.\times[R(R+\mu^2(\gamma^2-1))-2\gamma\mu^2(\gamma^2-1)Q]\right\}\\
\zeta_3 &= \delta_3+\beta_3+\frac{1}{2(1+\mu)^3}\left\{-(2+\mu(3+\gamma))+\frac{\mu}{S}[(4+3\mu-\mu\gamma)(9+\gamma+2\mu+2\mu\gamma)\right.\\
&-8(1+\mu)(\gamma-1)+\mu^2(\gamma^2-1)(7+5\gamma+3\mu+3\mu\gamma)]\\
&\left.-\frac{2\mu^2(\gamma+1)}{S^2}[R(R+\mu^2(\gamma^2-1))-2\gamma\mu^2(\gamma^2-1)Q]\right\}\\
\zeta_4 &= \delta_4+\beta_4-\frac{1}{(1+\mu)^2}\left\{\mu+\frac{\mu R}{S}-\frac{\mu^2}{S^2}[(R+\mu^2(\gamma^2-1))R-2\gamma\mu^2(\gamma^2-1)Q]\right\}\\
\zeta_5 &= \delta_5+\beta_5+\frac{4-\mu(1+\gamma)}{2Q(1+\mu)^2}\\
\zeta_6 &= \delta_6+\beta_6+\frac{(\gamma+1)}{4(1+\mu)^3Q^2}\left\{20+\mu(3-5\gamma)+\mu^2(\gamma^2+4\gamma-9)-\mu^3(\gamma^2-1)\right\}\\
\zeta_7 &= \frac{\mu(\gamma+1)}{2(1+\mu)^3}
\end{aligned}
$$

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