

# 3-D KINEMATICAL CONSERVATION LAWS (KCL): EQUATIONS OF EVOLUTION OF A SURFACE

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ABSTRACT. 3-D KCL are equations of evolution of a **propagating** surface  $\Omega_t$  in 3-space dimensions and were first derived by Giles, Prasad and Ravindran in 1995 assuming the motion of the surface to be isotropic. Here we discuss various properties of these 3-D KCL. These are the most general equations in conservation form, governing the evolution of  $\Omega_t$  with singularities which we call **kinks** and which are curves across which the normal  $\mathbf{n}$  to  $\Omega_t$  and amplitude  $w$  on  $\Omega_t$  are discontinuous. From KCL we derive a system of six differential equations and show that the KCL system is equivalent to the ray equations for  $\Omega_t$ . The six independent equations and an energy transport equation (for small amplitude waves in a polytropic gas) involving an amplitude  $w$  (which is related to the normal velocity  $m$  of  $\Omega_t$ ) form a completely determined system of seven equations. We have determined eigenvalues of the system by a very novel method and find that the system has two distinct nonzero eigenvalues and five zero eigenvalues and the dimension of the eigenspace associated with the multiple eigenvalue 0 is only 4. For an appropriately defined  $m$ , the two nonzero eigenvalues are real when  $m > 1$  and pure imaginary when  $m < 1$ . Finally we have shown how to use the theory to concrete examples.

**The symbols used in this paper are listed in the Appendix E**

## 1. INTRODUCTION

Propagation of a nonlinear wavefront and a shock front in three dimensional space  $\mathbb{R}^3$  are very complex physical phenomena and both fronts share a common property of possessing curves of discontinuities across which the normal direction to the fronts and the amplitude distribution on them suffer discontinuities. These are discontinuities of the first kind, i.e., the limiting values of the discontinuous functions and their derivatives on a front as we approach a curve of discontinuity from either side are finite. Such a discontinuity was first analyzed by Whitham in 1957 (see [18]), who called it shock-shock, meaning shock on a shock front. However, a discontinuity of this type is geometric in nature and can arise on any propagating surface  $\Omega_t$ , and we give it a general name **kink**. In order to explain the existence of a kink and study its formation and propagation, we need the governing equations in the form a system of physically realistic conservation laws. In this paper we derive and analyze such conservation laws in a specially defined *ray coordinate system* and since they are derived purely on geometrical consideration and we call them *kinematical conservation laws (KCL)*. When a discontinuous solution of the KCL

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*Date:* May 21, 2008.

*2000 Mathematics Subject Classification.* Primary 34A26, 35L65, 35L67; Secondary 35L80, 58J47.

*Key words and phrases.* Ray theory, kinematical conservation laws, nonlinear waves, conservation laws, shock propagation, curved shock, hyperbolic and elliptic systems, Fermat's principle.

system in the ray coordinates has a shock satisfying Rankine - Hugoniot conditions, the image of the shock in  $\mathbb{R}^3$  is a kink.

Before we start any discussion, we assume that all variables, both dependent and independent, used in this paper are nondimensional. There is one exception, the dependent variables in the first paragraph in section 4 are dimensional.

KCL governing the evolution of a moving curve  $\Omega_t$  in two space dimensions  $(x_1, x_2)$  were first derived by Morton, Prasad and Ravindran in 1992 [12], and the kink (in this case, a point on  $\Omega_t$ ) phenomenon is well understood (see [13]-section 3.3). We call this system of KCL as 2-D KCL which we describe in the next paragraph.

Consider a one parameter family of curves  $\Omega_t$  in  $(x_1, x_2)$ -plane, where the subscript  $t$  is the parameter whose different values give different positions of a moving curve. We assume that this family of curves has been obtained with the help of a ray velocity  $\boldsymbol{\chi} = (\chi_1, \chi_2)$ , which is a function of  $x_1, x_2, t$  and  $\mathbf{n}$ , where  $\mathbf{n}$  is the unit normal to  $\Omega_t$ . We assume that motion of this curve  $\Omega_t$  is isotropic so that we take the ray velocity  $\boldsymbol{\chi}$  in the direction of  $\mathbf{n}$  and write it as

$$(1.1) \quad \boldsymbol{\chi} = m\mathbf{n},$$

where the scalar function  $m$  depends on  $\mathbf{x}$  and  $t$  but is independent of  $\mathbf{n}$ . The ray equations

$$(1.2) \quad \frac{d\mathbf{x}}{dt} = m\mathbf{n}, \quad \frac{d\theta}{dt} = - \left( -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2} \right) m,$$

where  $\mathbf{n} = (n_1, n_2) = (\cos \theta, \sin \theta)$  are derived from the Charpit's equations (or Hamilton's canonical equations) of the eikonal equation (see section 2). The normal velocity  $m$  of  $\Omega_t$  is a nondimensionlized velocity with respect to a characteristic velocity (say the sound velocity  $a_0$  in a uniform ambient medium in the case  $\Omega_t$  is a wavefront in such a medium). Given a representation of the curve  $\Omega_0$  at the time  $t = 0$  in the form  $\mathbf{x} = \mathbf{x}_0(\xi)$ , we determine the unit normal  $\mathbf{n}_0(\xi)$  and then we solve the system (1.2) with these as initial values (this is a simplified view - the system (1.2) is usually underdetermined as explained below). Thus we get a representation of the curve  $\Omega_t$  at time  $t$  in the form  $\mathbf{x} = \mathbf{x}(\xi, t)$ . We assume (for development of the theory) that this gives a mapping:  $(\xi, t) \rightarrow (x_1, x_2)$  which is one to one. In this way we have introduced a ray coordinate system  $(\xi, t)$  such that  $t = \text{constant}$  represents the curve  $\Omega_t$  and  $\xi = \text{constant}$  represents a ray. Then  $m dt$  is an element of distance along a ray, i.e.,  $m$  is the metric associated with the variable  $t$ . Let  $g$  be the metric associated with the variable  $\xi$ , then

$$(1.3) \quad \frac{1}{g} \frac{\partial}{\partial \xi} = -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2}.$$

Simple, geometrical consideration gives (see [13]-section 3.3 and also the section 3 of this paper)

$$(1.4) \quad d\mathbf{x} = (g\mathbf{u})d\xi + (m\mathbf{n})dt,$$

where  $\mathbf{u}$  is the tangent vector to  $\Omega_t$ , i.e.,  $\mathbf{u} = (-n_2, n_1)$ . Equating  $(x_1)_{\xi t} = (x_1)_{t\xi}$  and  $(x_2)_{\xi t} = (x_2)_{t\xi}$ , we get the 2-D KCL

$$(1.5) \quad (gn_2)_t + (mn_1)_\xi = 0, \quad (gn_1)_t - (mn_2)_\xi = 0.$$

Using these KCL we can derive the Rankine-Hugonit conditions (i.e., the jump relations) relating the quantities on the two sides of a shock path in  $(\xi, t)$ -plane or a

kink path in  $(x_1, x_2)$ -plane. The system (1.5) is underdetermined since it contains only two equations in three variables  $\theta, m$  and  $g$ . It is possible to close it in many ways. One possible way is to close it by a single conservation law

$$(1.6) \quad (gG^{-1}(m))_t = 0,$$

where  $G$  is a given function of  $m$ . Baskar and Prasad [3] have studied the Riemann problem for the system (1.5) or (1.6) assuming some physically realistic conditions on  $G(m)$ . For a weakly nonlinear wavefront ([13]-chapter 6) in a polytropic gas, conservation of energy along a ray tube gives (with a suitable choice of  $\xi$ )

$$(1.7) \quad G(m) = (m - 1)^{-2} e^{-2(m-1)},$$

(see also the equation (6.6) in this article). Prasad and his collaborators have used this closure relation to solve many interesting problems and obtained many new results [4, 6, 11, 16]. KCL with (1.6) and (1.7) is a very interesting system. It is hyperbolic for  $m > 1$  and has elliptic nature for  $m < 1$ .

In this article, we shall discuss an extension of 2-D KCL to 3-D KCL. We start with a review of the ray theory in section 2. A derivation of 3-D KCL of Giles, Prasad and Ravindran (GPR) [9] is given in section 3. The derivation requires a pair of coordinates  $(\xi_1, \xi_2)$  on  $\Omega_t$  at any time  $t$ . The coordinates  $(\xi_1, \xi_2)$  evolve according to geometrical laws (i.e., KCL) and dynamical laws of the medium in which  $\Omega_t$  propagates. In section 4 we give an explicit differential form of the KCL and in section 5 we show its equivalence to the ray equations. In section 6, we derive a conservation form of the energy transport equation along rays for a small amplitude waves in a polytropic gas and then we close the 3-D KCL by this energy transport equation. We call the system of 7 conservation laws, six KCL and the energy transport equation, the equations of weakly nonlinear ray theory (WNLRT). We have two systems of equations in differential form: system-I consists of two of the ray equations, which are equations for first two components  $n_1$  and  $n_2$  of  $\mathbf{n}$  and the energy transport equation; and system-II consists of seven differential forms of the equations of WNLRT (i.e., the KCL and the energy transport equation). In section 7, we discuss the eigenvalues and eigenvectors of the system-I and in section 8 we do that for the system-II. In section 8.4, we derive the nonzero eigenvalues of the system-I from those of the system-II and vice versa. This article, therefore, puts the theory of 3-D KCL on a strong foundation and the theory can be used to discuss evolution of a surface  $\Omega_t$  in 3-space dimensions and formation and propagation of curves of singularities on  $\Omega_t$ . In section 9 we indicate how to choose the ray coordinates for a particular problem and set up an initial value problem for KCL. Arun, Lukáčová and Rao [1] has successfully used the 3-D KCL theory developed here and obtained some very interesting results numerically. They have also raised some interesting mathematical questions regarding existence of  $\delta$ -waves in the solutions, which we plan to study in future.

## 2. A BRIEF DISCUSSION OF THE RAY EQUATIONS OF AN ISOTROPICALLY EVOLVING FRONT $\Omega_t$

Though it is possible to derive KCL for a more general motion of a moving surface  $\Omega_t$  (following [13] for 2-D KCL), we consider here only to the case when the motion of  $\Omega_t$  is isotropic in the sense that the associated ray velocity  $\chi$  depends on

the unit normal  $\mathbf{n}$  by the relation (1.1). An example of this is the wave equation

$$(2.1) \quad u_{tt} - m^2 (u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}) = 0,$$

where  $m$  need not be constant. For this equation, we shall take only a forward facing wavefront  $\Omega_t$ , so that the associated characteristic surface  $\Omega$  in  $(\mathbf{x}, t)$ -space, given by  $\varphi(\mathbf{x}, t) = 0$ , satisfies the eikonal equation

$$(2.2) \quad \varphi_t + m \{ \varphi_{x_1}^2 + \varphi_{x_2}^2 + \varphi_{x_3}^2 \}^{1/2} = 0.$$

Note that  $\Omega$  is a surface in space-time (i.e.,  $\mathbb{R}^4$ ) and  $\Omega_t$  given by  $\varphi(\mathbf{x}, t) = 0$ , for  $t = \text{constant}$  is a surface in 3-D  $\mathbf{x}$ -space. For  $m$  as a given function of  $\mathbf{x}$  and  $t$ , bicharacteristic equations or the ray equations, ([13]-sections 2.4, 6.1, [14]) are

$$(2.3) \quad \frac{d\mathbf{x}}{dt} = m\mathbf{n}, \quad |\mathbf{n}| = 1,$$

$$(2.4) \quad \frac{d\mathbf{n}}{dt} = -\mathbf{L}m := -(\nabla - \mathbf{n}\langle \mathbf{n}, \nabla \rangle)m.$$

The bicharacteristics in  $(\mathbf{x}, t)$ -space form a 5 parameter family of curves. Now, we take a characteristic surface  $\Omega$  and note that its level set at  $t = 0$ , i.e., the surface  $\Omega_0$ :  $\varphi(\mathbf{x}, 0) = 0$  in the  $\mathbf{x}$ -space is a two dimensional manifold. Thus  $\Omega_0$  is represented parametrically as  $\mathbf{x} = \mathbf{x}_0(\xi_1, \xi_2)$ , from which the unit normal  $\mathbf{n}_0(\xi_1, \xi_2)$  of  $\Omega_0$  can be calculated. Now the bicharacteristics, which generate  $\Omega$  can be obtained by solving the equations (2.3) and (2.4) with initial data

$$(2.5) \quad \mathbf{x}|_{t=0} = \mathbf{x}_0(\xi_1, \xi_2) \text{ and } \mathbf{n}|_{t=0} = \mathbf{n}_0(\xi_1, \xi_2).$$

*Thus the bicharacteristic curves which generate a given characteristic surface  $\Omega$  in space-time form a two parameter family*

$$(2.6) \quad \mathbf{x} = \mathbf{x}(\mathbf{x}_0(\xi_1, \xi_2), t).$$

The rays, starting from the various points of  $\Omega_0$  are projections on  $\mathbf{x}$ -space of the above bicharacteristic curves. Fermat's method of construction of the wavefront  $\Omega_t$  at any time  $t$  consists of generating the surface  $\Omega_t$  from the solution (2.6) by keeping  $t$  constant and varying  $\xi_1$  and  $\xi_2$ . A front  $\Omega_t$  having an isotropic motion need not come from the wave equation. An example of this is the crest line of a curved solitary wave on the surface of a shallow water [2]. However, every isotropically evolving wavefront would satisfy an eikonal equation (2.2) with a suitable front velocity  $m$ . Evolution of such a front  $\Omega_t$  is given by the ray equations (2.3)-(2.4).

### 3. 3-D KCL OF GILES, PRASAD AND RAVINDRAN (1995)

Following the discussion in the last section consider a surface  $\Omega$  in  $\mathbb{R}^4$ ,  $\Omega$ :  $\varphi(\mathbf{x}, t) = 0$  and let us assume that  $\Omega$  is generated by a two parameter family of curves in  $\mathbb{R}^4$ , such that projection of these curves on  $\mathbf{x}$ -space are rays which are orthogonal to the successive position of the front  $\Omega_t$ :  $\varphi(\mathbf{x}, t) = 0, t = \text{constant}$ .

We introduce a ray coordinate system  $(\xi_1, \xi_2, t)$  in  $\mathbf{x}$ -space such that  $t = \text{constant}$  represents the surface  $\Omega_t$ , see [10]. The surface  $\Omega_t$  in  $\mathbf{x}$ -space is now generated by a one parameter family of curves such that along each of these curves  $\xi_1$  varies and the parameter  $\xi_2$  is constant. Similarly  $\Omega_t$  is generated by another one parameter family of curves along each of these  $\xi_2$  varies and  $\xi_1$  is constant. Through each point  $(\xi_1, \xi_2)$  of  $\Omega_t$  there passes a ray orthogonal (in  $\mathbf{x}$ -space) to the successive positions of  $\Omega_t$ , thus rays form a two parameter family as mentioned above. Given  $\xi_1, \xi_2$  and  $t$ , we uniquely identify a point  $P$  in  $\mathbf{x}$ -space. For the development of theory, we

assume that the mapping from  $(\xi_1, \xi_2, t)$ -space to  $(x_1, x_2, x_3)$ -space is one to one. On  $\Omega_t$  let  $\mathbf{u}$  and  $\mathbf{v}$  be unit tangent vectors of the curves  $\xi_2 = \text{constant}$  and  $\xi_1 = \text{constant}$  respectively and  $\mathbf{n}$  be unit normal to  $\Omega_t$ . Then

$$(3.1) \quad \mathbf{n} = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}.$$

Let an element of length along a curve ( $\xi_2 = \text{constant}$ ,  $t = \text{constant}$ ) be  $g_1 d\xi_1$  and that along a curve ( $\xi_1 = \text{constant}$ ,  $t = \text{constant}$ ) be  $g_2 d\xi_2$ . The element of length along a ray ( $\xi_1 = \text{constant}$ ,  $\xi_2 = \text{constant}$ ) is  $mdt$ . The displacement  $d\mathbf{x}$  in  $\mathbf{x}$ -space due to increments  $d\xi_1$ ,  $d\xi_2$  and  $dt$  is given by (this is an extension of the result (1.4))

$$(3.2) \quad d\mathbf{x} = (g_1 \mathbf{u})d\xi_1 + (g_2 \mathbf{v})d\xi_2 + (m\mathbf{n})dt.$$

This gives

$$(3.3) \quad J := \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, t)} = g_1 g_2 m \sin \chi, \quad 0 < \chi < \pi,$$

where  $\chi(\xi_1, \xi_2, t)$  is the angle between the  $\mathbf{u}$  and  $\mathbf{v}$ , i.e.,

$$(3.4) \quad \cos \chi = \langle \mathbf{u}, \mathbf{v} \rangle.$$

As explained after (4.5) and (6.6) in the next section, we shall like to choose  $\sin \chi = |\mathbf{u} \times \mathbf{v}|$  which requires the restriction  $0 < \chi < \pi$  on  $\chi$ . For a smooth moving surface  $\Omega_t$ , we equate  $\mathbf{x}_{\xi_1 t} = \mathbf{x}_{t\xi_1}$  and  $\mathbf{x}_{\xi_2 t} = \mathbf{x}_{t\xi_2}$ , and get the 3-D KCL of Giles, Prasad and Ravindran [9],

$$(3.5) \quad (g_1 \mathbf{u})_t - (m\mathbf{n})_{\xi_1} = 0,$$

$$(3.6) \quad (g_2 \mathbf{v})_t - (m\mathbf{n})_{\xi_2} = 0.$$

We also equate  $\mathbf{x}_{\xi_1 \xi_2} = \mathbf{x}_{\xi_2 \xi_1}$  and derive 3 more scalar equations contained in

$$(3.7) \quad (g_2 \mathbf{v})_{\xi_1} - (g_1 \mathbf{u})_{\xi_2} = 0.$$

Equations (3.5)-(3.7) are necessary and sufficient conditions for the integrability of the equation (3.2) (see [8], section 1.9).

From the equations (3.5) and (3.6) we can show that  $(g_2 \mathbf{v})_{\xi_1} - (g_1 \mathbf{u})_{\xi_2}$  does not depend on  $t$ . If any choice of coordinates  $\xi_1$  and  $\xi_2$  on  $\Omega_0$  implies that the condition (3.7) is satisfied at  $t = 0$  then it follows that (3.7) is automatically satisfied. Thus, the 3-D KCL is a system of six scalar evolution equations (3.5) and (3.6). However, since  $|\mathbf{u}| = 1$ ,  $|\mathbf{v}| = 1$ , there are 7 dependent variables in (3.5) and (3.6): two independent components of each of  $\mathbf{u}$  and  $\mathbf{v}$ , the front velocity  $m$  of  $\Omega_t$ ,  $g_1$  and  $g_2$ . Thus KCL is an underdetermined system and can be closed only with the help of additional relations or equations, which would follow from the nature of the surface  $\Omega_t$  and the dynamics of the medium in which it propagates.

We derive a few results from (3.5) and (3.6) without considering the closure equation (or equations) for  $m$ . The system (3.5) and (3.6) consists of equations which are conservation laws, so its weak solution may contain shocks which are surfaces in  $(\xi_1, \xi_2, t)$ -space. Across these **shock surfaces**  $m, g_1, g_2$  and vectors  $\mathbf{u}, \mathbf{v}$  and  $\mathbf{n}$  will be discontinuous. Image of a shock surface into  $\mathbf{x}$ -space will be another surface, let us call it a **kink surface**, which will intersect  $\Omega_t$  in a curve, say **kink curve**  $\mathcal{K}_t$ . Across this kink curve or simply the kink, the normal direction  $\mathbf{n}$  of  $\Omega_t$  will be discontinuous as shown in Figure 1. As time  $t$  evolves,  $\mathcal{K}_t$  will generate the kink surface. A **shock front** (a phrase very commonly used in literature) is a curve

in  $(\xi_1, \xi_2)$ -plane and its motion as  $t$  changes generates the shock surface in  $(\xi_1, \xi_2, t)$ -space. We assume that the mapping between  $(\xi_1, \xi_2, t)$ -space and  $(x_1, x_2, x_3)$ -space continues to be one to one even when a kink appears.

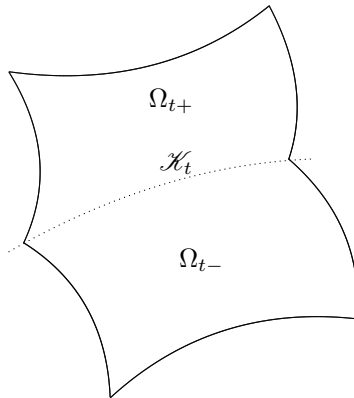


FIGURE 1. Kink curve  $\mathcal{H}_t$  (shown with dotted lines) on  $\Omega_t = \Omega_{t+} \cup \Omega_{t-}$

The distance  $d\mathbf{x}$  between two points  $P(\mathbf{x})$  and  $Q'(\mathbf{x} + d\mathbf{x})$  on  $\Omega_t$  and  $\Omega_{t+dt}$  respectively satisfies the relation (3.2), where  $(\xi_1, \xi_2, t)$  and  $(\xi_1 + d\xi_1, \xi_2 + d\xi_2, t + dt)$  are corresponding coordinates in  $(\xi_1, \xi_2, t)$ -space. If the points  $P$  and  $Q'$  are chosen to be points on the kink surface (see [13] for a two dimensional analog), then the conservation of  $d\mathbf{x}$  implies that the expression for  $(d\mathbf{x})_+$  on one side of the kink surface must be equal to the expression for  $(d\mathbf{x})_-$  on the other side. Denoting quantities on the two sides of the kink by subscripts  $+$  and  $-$ , we get

$$(3.8) \quad \begin{aligned} g_{1+}d\xi_1\mathbf{u}_+ + g_{2+}d\xi_2\mathbf{v}_+ + m_+d\mathbf{t}\mathbf{n}_+ \\ = g_{1-}d\xi_1\mathbf{u}_- + g_{2-}d\xi_2\mathbf{v}_- + m_-d\mathbf{t}\mathbf{n}_-. \end{aligned}$$

We take the direction of the line element  $PQ'$  such that its projection on  $(\xi_1, \xi_2)$ -plane is in the direction of the normal to the shock curve in  $(\xi_1, \xi_2)$ -plane, then the differentials are further restricted. Let the unit normal of this shock curve be  $(E_1, E_2)$  and let  $K$  be its velocity of propagation in this plane, then the differentials in (3.8) satisfy  $\frac{d\xi_1}{dt} = E_1K$  and  $\frac{d\xi_2}{dt} = E_2K$ , and (3.8) now becomes

$$(3.9) \quad \begin{aligned} (g_{1+}E_1\mathbf{u}_+ + g_{2+}E_2\mathbf{v}_+)K + m_+\mathbf{n}_+ \\ = (g_{1-}E_1\mathbf{u}_- + g_{2-}E_2\mathbf{v}_-)K + m_-\mathbf{n}_-. \end{aligned}$$

Thus (3.9) is a condition for the conservation of distance (in three independent directions in  $\mathbf{x}$ -space) across a kink surface when a point moves along the normal to the shock curve in  $(\xi_1, \xi_2)$ -plane.

Using the usual method for the derivation of jump conditions across a shock, we deduce the from conservation laws (3.5) and (3.6)

$$(3.10) \quad K[g_1\mathbf{u}] + E_1[m\mathbf{n}] = 0, \quad K[g_2\mathbf{v}] + E_2[m\mathbf{n}] = 0,$$

where a jump  $[f]$  of a quantity  $f$  is defined by

$$(3.11) \quad [f] = f_+ - f_-.$$

Multiply the first relation in (3.10) by  $E_1$  and the second relation by  $E_2$ , adding and using  $E_1^2 + E_2^2 = 1$ , we get

$$(3.12) \quad E_1 K[g_1 \mathbf{u}] + E_2 K[g_2 \mathbf{v}] + [m\mathbf{n}] = 0.$$

which is the same as (3.9). Thus we have proved a theorem of GPR, [9].

**Theorem 3.1.** *The six jump relations (3.10) imply conservation of distance in  $x_1, x_2$  and  $x_3$  directions (and hence in any arbitrary direction in  $\mathbf{x}$ -space) in the sense that the expressions for a vector displacement  $(d\mathbf{x})_{\mathcal{K}_t}$  of a point of the kink line  $\mathcal{K}_t$  in an infinitesimal time interval  $dt$ , when computed in terms of variables on the two sides of a kink surface, have the same value. This displacement of the point is assumed to take place on the kink surface and that of its image in  $(\xi_1, \xi_2, t)$ -space takes place on the shock surface such that the corresponding displacement in  $(\xi_1, \xi_2)$ -plane is with the shock front (i.e., it is in direction  $\frac{d}{d\xi}(\xi_1, \xi_2) = (E_1, E_2)K$ ).*

This theorem assures that the 3-D KCL are physically realistic.

Consider a point  $P$  on a kink line  $\mathcal{K}_t$  on  $\Omega_t$  and two straight lines  $T_-$  and  $T_+$  orthogonal to the kink line at  $P$  and lying in the tangent planes at  $P$  to  $\Omega_{t-}$  and  $\Omega_{t+}$  on the two sides of  $\mathcal{K}_t$ . Let  $N_-$  and  $N_+$  be normals to the two tangent planes at  $P$ . Then the four lines  $T_+, N_+, N_-$  and  $T_-$ , being orthogonal to the kink line at  $P$ , are coplanar. A kink phenomenon is basically two dimensional. Locally, the two sides  $\Omega_{t-}$  and  $\Omega_{t+}$  of  $\Omega_t$  can be regarded to be planes separated by a straight kink line. Hence the evolution of the kink phenomena can be viewed locally in a plane which intersects the planes  $\Omega_{t-}, \Omega_{t+}$  and  $\mathcal{K}_t$  orthogonally as shown in the Figure 3.3.4 of [13].

We state an important result which will be very useful in proving many properties of the KCL. *Let  $P_0(\mathbf{x}_0)$  be a given point on  $\Omega_t$ . Then there exist two one parameter families of smooth curves on  $\Omega_t$  such that the unit vectors  $\mathbf{u}_0$  and  $\mathbf{v}_0$  along the members of the curves through the chosen point  $P_0$  can have any two arbitrary directions and the metrics  $g_{10}$  and  $g_{20}$  at this point can have any two positive values.*

#### 4. AN EXPLICIT DIFFERENTIAL FORM OF KCL

Writing the differential form of (3.5) and taking inner product with  $\mathbf{u}$  and using  $\langle \mathbf{u}, \mathbf{n}_{\xi_1} \rangle = -\langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle$  we get

$$(4.1) \quad g_{1t} = -m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle.$$

Similarly,

$$(4.2) \quad g_{2t} = -m \langle \mathbf{n}, \mathbf{v}_{\xi_2} \rangle.$$

In the differential form of (3.5), we use the expression (4.1) for  $g_{1t}$  and get

$$(4.3) \quad g_1 \mathbf{u}_t = m_{\xi_1} \mathbf{n} + m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle \mathbf{u} + m \mathbf{n}_{\xi_1}.$$

Similarly

$$(4.4) \quad g_2 \mathbf{v}_t = m_{\xi_2} \mathbf{n} + m \langle \mathbf{n}, \mathbf{v}_{\xi_2} \rangle \mathbf{v} + m \mathbf{n}_{\xi_2}.$$

In order that

$$(4.5) \quad \cos \chi = \langle \mathbf{u}, \mathbf{v} \rangle \text{ and } \sin \chi = |\mathbf{u} \times \mathbf{v}|$$

is valid, we choose  $\chi$  the angle between  $\mathbf{u}$  and  $\mathbf{v}$  to satisfy  $0 < \chi < \pi$ . Then

$$(4.6) \quad \begin{aligned} |\mathbf{u} \times \mathbf{v}|_{\xi_1} &= (\sin \chi)_{\xi_1} = -\frac{\cos \chi}{\sin \chi} (\cos \chi)_{\xi_1} \\ &= -\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u} \times \mathbf{v}|} \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1}. \end{aligned}$$

Hence, from (3.1)

$$(4.7) \quad \mathbf{n}_{\xi_1} = \frac{1}{|\mathbf{u} \times \mathbf{v}|} \left\{ (\mathbf{u} \times \mathbf{v})_{\xi_1} + \frac{\mathbf{n} \langle \mathbf{u} \times \mathbf{v} \rangle}{|\mathbf{u} \times \mathbf{v}|} \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1} \right\}.$$

Substituting the expressions (3.1) for  $\mathbf{n}$  and (4.7) for  $\mathbf{n}_{\xi_1}$  in (4.3) we get a form of an equation for  $\mathbf{u}$  in which  $\mathbf{u}_t$  expressed purely in terms of  $m, g_1, \mathbf{u}$  and  $\mathbf{v}$ . Similarly, we can get a form of an equation for  $\mathbf{v}$ . Now we assert<sup>2</sup> that the third scalar equation in (4.3), i.e., the equation for  $u_3$  (or in (4.4), i.e., the equation for  $v_3$ ) can be derived from the first two equations in (4.3) (or in (4.4)). Thus, (4.1)-(4.4) contain a set of 6 independent differential forms of equations of the KCL (3.5) and (3.6) purely in terms of  $m, g_1, \mathbf{u}$  and  $\mathbf{v}$ .

We now proceed to derive an explicit form of the equations (4.1)-(4.2) and the first two in each of the two equations (4.3) and (4.4), i.e., we write these equations in terms of variables  $g_1, g_2, u_1, u_2, v_1$  and  $v_2$  only (i.e., free from  $u_3$  and  $v_3$ ). This involves long calculations. We first express derivatives of  $u_3$  in terms of those of  $u_1$  and  $u_2$  using the relation  $u_3^2 = 1 - u_1^2 - u_2^2$ . This immediately leads to equations for  $g_1$  and  $g_2$  in the forms

$$(4.8) \quad g_{1t} - m \frac{n_3 u_1 - n_1 u_3}{u_3} u_{1\xi_1} + m \frac{n_2 u_3 - n_3 u_2}{u_3} u_{2\xi_1} = 0,$$

$$(4.9) \quad g_{2t} + m \frac{n_1 v_3 - n_3 v_1}{v_3} v_{1\xi_2} - m \frac{n_3 v_2 - n_2 v_3}{v_3} v_{2\xi_2} = 0.$$

We take  $(\mathbf{u}, \mathbf{v}, \mathbf{n})$  to be a right handed set of vectors, then

$$(4.10) \quad (n_1, n_2, n_3) = \frac{1}{\sin \chi} (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Using these expressions for components of  $\mathbf{n}$  and using  $u_1^2 + u_2^2 + u_3^2 = 1$  and  $u_1 v_1 + u_2 v_2 + u_3 v_3 = \cos \chi$  we get the following expressions for some terms in the coefficients in (4.8)

$$(4.11) \quad n_3 u_1 - n_1 u_3 = \frac{v_2}{\sin \chi} - u_2 \cot \chi, \quad n_2 u_3 - n_3 u_2 = \frac{v_1}{\sin \chi} - u \cot \chi.$$

We can do similar calculation for the coefficients in equation (4.9). Thus, we get the required equations for  $g_1$  and  $g_2$

$$(4.12) \quad g_{1t} - m \frac{v_2 - u_2 \cos \chi}{u_3 \sin \chi} u_{1\xi_1} + m \frac{v_1 - u_1 \cos \chi}{u_3 \sin \chi} u_{2\xi_1} = 0,$$

$$(4.13) \quad g_{2t} + m \frac{u_2 - v_2 \cos \chi}{v_3 \sin \chi} v_{1\xi_2} - m \frac{u_1 - v_1 \cos \chi}{u_3 \sin \chi} v_{2\xi_2} = 0.$$

When we use (4.6) in the equation (4.3) for  $u_1$ , we note that we need to find expressions for  $u_{3\xi_1}$ , and the first components of  $(\mathbf{u} \times \mathbf{v})_{\xi_1}$  and  $\mathbf{n} \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1}$

<sup>2</sup>See Lemma A.1 and its proof in the Appendix A

expressed purely in terms of  $u_1, u_2, v_1$  and  $v_2$ . This requires long calculations which we present in Appendix B. The calculations leads to

$$(4.14) \quad g_1 u_{1t} = n_1 m_{\xi_1} + m \left\{ \frac{u_1 u_2 + n_1 n_2}{u_3 \sin \chi} \cos \chi u_{1\xi_1} - \frac{u_1^2 + n_1^2 - 1}{u_3 \sin \chi} \cos \chi u_{2\xi_1} \right. \\ \left. - \frac{u_2 v_1 + n_1 n_2 \cos \chi}{v_3 \sin \chi} v_{1\xi_1} + \frac{u_1 v_1 + (n_1^2 - 1) \cos \chi}{v_3 \sin \chi} v_{2\xi_1} \right\}$$

which is of the desired form since  $u_3, v_3$  and components of  $\mathbf{n}$  can be expressed in terms of  $u_1, u_2, v_1$  and  $v_2$ . Similarly, we can find equations of evolution for  $u_2, v_1$  and  $v_2$ . Collecting all these results, we get the following explicit differential forms of the equations for  $u_1, u_2, v_1$  and  $v_2$ . In these equations derivatives of  $\mathbf{n}, u_3$  and  $v_3$  do not appear.

$$(4.15) \quad g_1 u_{1t} - n_1 m_{\xi_1} + b_{11}^{(1)} u_{1\xi_1} + b_{12}^{(1)} u_{2\xi_1} + b_{13}^{(1)} v_{1\xi_1} + b_{14}^{(1)} v_{2\xi_1} = 0,$$

$$(4.16) \quad g_1 u_{2t} - n_2 m_{\xi_1} + b_{21}^{(1)} u_{1\xi_1} + b_{22}^{(1)} u_{2\xi_1} + b_{23}^{(1)} v_{1\xi_1} + b_{24}^{(1)} v_{2\xi_1} = 0,$$

$$(4.17) \quad g_2 v_{1t} - n_1 m_{\xi_2} + b_{31}^{(2)} u_{1\xi_2} + b_{32}^{(2)} u_{2\xi_2} + b_{33}^{(2)} v_{1\xi_2} + b_{34}^{(2)} v_{2\xi_2} = 0,$$

$$(4.18) \quad g_2 v_{2t} - n_2 m_{\xi_2} + b_{41}^{(2)} u_{1\xi_2} + b_{42}^{(2)} u_{2\xi_2} + b_{43}^{(2)} v_{1\xi_2} + b_{44}^{(2)} v_{2\xi_2} = 0,$$

where the coefficients  $b_{ij}^{(1)}$  and  $b_{ij}^{(2)}$  are given in the Appendix B.

We find a very interesting between the coefficients  $b_{ij}^{(1)}$  and  $b_{ij}^{(2)}$ . This is obtained as a consistency condition between two different expressions for  $m_{\xi_1}$  and  $m_{\xi_2}$ . We proceed to derive the expressions for  $m_{\xi_1}$ . Differentiate  $m^2 = m^2(n_1^2 + n_2^2 + n_3^2)$  to derive

$$m_{\xi_1} = n_1(mn_1)_{\xi_1} + n_2(mn_2)_{\xi_1} + n_3(mn_3)_{\xi_1},$$

use (2.3) for the expressions in the brackets and interchange the order of derivatives to get

$$(4.19) \quad m_{\xi_1} = \langle \mathbf{n}, (\mathbf{x}_{\xi_1})_t \rangle = \langle \mathbf{n}, (g_1 \mathbf{u})_t \rangle \\ = g_{1t} \langle \mathbf{n}, \mathbf{u} \rangle + g_1 \langle \mathbf{n}, \mathbf{u}_t \rangle = g_1 \langle \mathbf{n}, \mathbf{u}_t \rangle \\ = g_1 \alpha_1 u_{1t} + g_1 \alpha_2 u_{2t}, \quad \text{after using } u_3^2 = 1 - u_1^2 - u_2^2,$$

where

$$(4.20) \quad \alpha_1 = -\frac{n_3 u_1 - n_1 u_3}{u_3} =: \frac{1}{m} b_{61}^{(1)}, \quad \alpha_2 = \frac{n_2 u_3 - n_3 u_2}{u_3} =: \frac{1}{m} b_{62}^{(1)}.$$

Using the expressions for  $u_{1t}$  and  $u_{2t}$  in (4.19) from (4.15) and (4.16), we get an identity. Equating coefficients  $m_{\xi_1}, u_{1\xi_1}, u_{2\xi_1}, v_{1\xi_1}, v_{2\xi_1}$  we get 5 consistency conditions

$$(4.21) \quad n_1 \alpha_1 + n_2 \alpha_2 = 1, \quad \alpha_1 b_{11}^{(1)} + \alpha_2 b_{21}^{(1)} = 0, \quad \alpha_1 b_{21}^{(1)} + \alpha_2 b_{22}^{(1)} = 0, \\ \alpha_1 b_{13}^{(1)} + \alpha_2 b_{23}^{(1)} = 0, \quad \alpha_1 b_{14}^{(1)} + \alpha_2 b_{24}^{(1)} = 0.$$

Similarly, starting from  $m_{\xi_2}$ , we get another set of consistency conditions in terms of  $\beta_3$  and  $\beta_4$ , where

$$(4.22) \quad \beta_3 = \frac{n_1 v_3 - n_3 v_1}{v_3} =: \frac{1}{m} b_{73}^{(2)}, \quad \beta_4 = -\frac{n_3 v_2 - n_2 v_3}{v_3} =: \frac{1}{m} b_{74}^{(2)}.$$

$$(4.23) \quad n_1 \beta_3 + n_2 \beta_4 = 1, \quad \beta_3 b_{31}^{(2)} + \beta_4 b_{41}^{(2)} = 0, \quad \beta_3 b_{32}^{(2)} + \beta_4 b_{42}^{(2)} = 0, \\ \beta_3 b_{33}^{(2)} + \beta_4 b_{43}^{(2)} = 0, \quad \beta_3 b_{34}^{(2)} + \beta_4 b_{44}^{(2)} = 0.$$

(4.21) and (4.23) show nice relations in the coefficients of the equations (4.15)-(4.18). We have used these relations to simplify numerical computation of eigenvalues in the section 8.2.

3D-KCL being only 6 equations in seven quantities  $u_1, u_2, v_1, v_2, m, g_1$  and  $g_2$ , it is an underdetermined system. This is expected as KCL is purely a mathematical result and the dynamics of a particular moving surface  $\Omega_t$  has played no role in the derivation of KCL. In our previous investigations, we have closed the 2-D KCL for three different types of  $\Omega_t$  ([2], [13]-chapters 6 and 10), one of them being the case when  $\Omega_t$  is a weakly nonlinear wavefront in a polytropic gas, which we shall consider again in the section 6 for 3D-KCL. We shall use the energy transport equation along rays of the weakly nonlinear ray theory in a polytropic gas to derive the closure equation in the form of an additional conservation law.

## 5. EQUIVALENCE OF KCL AND RAY EQUATIONS

Let us start with a given smooth function  $m$  of  $\mathbf{x}$  and  $t$  and let  $\mathbf{x}, \mathbf{n}$  (with  $|\mathbf{n}| = 1$ ) satisfy the ray equations (2.3) and (2.4), which give successive positions of a moving surface  $\Omega_t$ . Choose a coordinate system  $(\xi_1, \xi_2)$  on  $\Omega_t$  with metrics  $g_1$  and  $g_2$  associated with  $\xi_1$  and  $\xi_2$  respectively. Let  $\mathbf{u}$  and  $\mathbf{v}$  be unit tangent vectors along the curves  $\xi_2 = \text{constant}$  and  $\xi_1 = \text{constant}$  respectively. Then the derivation of the section 3 leads to the equations (3.5)-(3.7) and hence the 3-D KCL. Thus the ray equations imply 3-D KCL.

In addition to the above proof, let us give a direct derivation of the KCL equations (4.1) and (4.2) from the first ray equations, i.e, equation (2.3). Definition of the metric  $g_1$  gives  $g_1^2 = x_1^2_{\xi_1} + x_2^2_{\xi_1} + x_3^2_{\xi_1} = |\mathbf{x}_{\xi_1}|^2$ . Differentiating it with respect to  $t$ , using  $\mathbf{x}_{\xi_1 t} = \mathbf{x}_{t\xi_1}$  and  $\mathbf{x}_{\xi_1} = g_1 \mathbf{u}$  (from (3.2)) we get

$$(5.1) \quad g_{1t} = \langle \mathbf{u}, (\mathbf{x}_t)_{\xi_1} \rangle.$$

Using (2.3) in this

$$(5.2) \quad \begin{aligned} g_{1t} &= \langle \mathbf{u}, m_{\xi_1} \mathbf{n} + m \mathbf{n}_{\xi_1} \rangle \\ &= \langle \mathbf{u}, m \mathbf{n}_{\xi_1} \rangle = -m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle. \end{aligned}$$

which is the equation (4.1). Similarly the equation (4.2) can be derived.

Now we take up the proof of the converse, i.e., the derivation of the ray equations from the KCL (3.5)-(3.6). We are given three smooth unit vector fields  $\mathbf{u}, \mathbf{v}, \mathbf{n}$  and three smooth scalar functions  $m, g_1$  and  $g_2$  in  $(\xi_1, \xi_2, t)$ -space such that  $\mathbf{n}$  is orthogonal to  $\mathbf{u}$  and  $\mathbf{v}$ , i.e.,

$$(5.3) \quad \langle \mathbf{n}, \mathbf{u} \rangle = 0 \quad \text{and} \quad \langle \mathbf{n}, \mathbf{v} \rangle = 0$$

and they satisfy the KCL (3.5)-(3.7).

According to the fundamental integrability theorem ([8]-page 104), the conditions (3.5)-(3.7) imply the existence of a vector  $\mathbf{x}$  satisfying (3.2), i.e.,

$$(5.4) \quad (\mathbf{x}_t, \mathbf{x}_{\xi_1}, \mathbf{x}_{\xi_2}) = (m \mathbf{n}, g_1 \mathbf{u}, g_2 \mathbf{v}).$$

This gives a one to one mapping between  $\mathbf{x}$ -space and  $(\xi_1, \xi_2, t)$ -space as long as the Jacobian (3.3) is neither zero nor infinity. Let  $t = \text{constant}$  in  $(\xi_1, \xi_2, t)$ -space is mapped on to a surface  $\Omega_t$  in  $\mathbf{x}$ -space on which  $\xi_1$  and  $\xi_2$  are surface coordinates. Then  $\mathbf{u}$  and  $\mathbf{v}$  are tangent to  $\Omega_t$  and (5.3) show that  $\mathbf{n}$  is orthogonal to  $\Omega_t$ . Let  $\varphi(\mathbf{x}, t) = 0$  be the equation of  $\Omega_t$ , then  $\mathbf{n} = \nabla \varphi / |\nabla \varphi|$ . The relation  $\mathbf{x}_t = m \mathbf{n}$  in (5.4) is nothing but the first part of the ray equation and shows that  $m$  is the

normal velocity of  $\Omega_t$ . The function  $\varphi$  now satisfies the eikonal equation (2.2) which implies (2.4), see also [14]. Thus, we have derived the ray equations from KCL.

Now we have completed the proof of the theorem

**Theorem 5.1.** *For a given smooth function  $m$  of  $\mathbf{x}$  and  $t$ , the ray equations (2.3) and (2.4) are equivalent to the KCL as long as their solutions are smooth.*

Though we have established equivalence of two systems, it is instructive to derive the second part of the ray equations, i.e., the equations (2.4) from KCL (3.5)-(3.6) by direct calculation, which we do below.

Transformation (5.4) between  $\mathbf{x}$ -space and  $(\xi_1, \xi_2, t)$ -space, implies relations

$$(5.5) \quad \frac{\partial}{\partial t} = \langle m\mathbf{n}, \nabla \rangle, \quad \frac{\partial}{g_1 \partial \xi_1} = \langle \mathbf{u}, \nabla \rangle, \quad \frac{\partial}{g_2 \partial \xi_2} = \langle \mathbf{v}, \nabla \rangle$$

between the partial derivatives in  $(x, y, z)$  and  $(\xi_1, \xi_2, t)$  coordinates. Solving  $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$  in terms of  $\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial t}$ , we get

$$(5.6) \quad \nabla := \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = \frac{\mathbf{n}}{m} \frac{\partial}{\partial t} + \frac{\mathbf{u} - \mathbf{v} \cos \chi}{\sin \chi} \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \frac{\mathbf{v} - \mathbf{u} \cos \chi}{\sin \chi} \frac{1}{g_2} \frac{\partial}{\partial \xi_2}.$$

Differentiating the relations  $|\mathbf{n}| = 1, \langle \mathbf{n}, \mathbf{u} \rangle = 0$  and  $\langle \mathbf{n}, \mathbf{v} \rangle = 0$  with respect to  $t$ , and solving for  $n_{1t}$  and  $n_{2t}$ , we get

$$(5.7) \quad n_{1t} = \frac{(n_3 v_2 - n_2 v_3)}{u_3} \{ (n_3 u_1 - n_1 u_3) u_{1t} - (n_2 u_3 - n_3 u_2) u_{2t} \} \\ + \frac{(n_2 u_3 - n_3 u_2)}{v_3} \{ -(n_1 v_3 - n_3 v_1) v_{1t} + (n_3 v_2 - n_2 v_3) v_{2t} \}.$$

and

$$(5.8) \quad n_{2t} = \frac{(n_1 v_3 - n_3 v_1)}{u_3} \{ (n_3 u_1 - n_1 u_3) u_{1t} - (n_2 u_3 - n_3 u_2) u_{2t} \} \\ + \frac{(n_3 u_1 - n_1 u_3)}{v_3} \{ -(n_1 v_3 - n_3 v_1) v_{1t} + (n_3 v_2 - n_2 v_3) v_{2t} \}.$$

Now we substitute the expressions for  $u_{1t}, u_{2t}, v_{1t}$  and  $v_{2t}$  from (4.15)-(4.18) in the terms on the right hand side of above equations and after long calculations (see Appendix C), we find that all terms  $u_{i\xi_i}, v_{i\xi_i}, i = 1, 2$  drop out and only derivatives of  $m$  appear. The final equations are

$$(5.9) \quad \frac{\partial n_1}{\partial t} = -\frac{1}{\sin \chi} \left\{ (u_1 - v_1 \cos \chi) \frac{\partial m}{g_1 \partial \xi_1} + (v_1 - u_1 \cos \chi) \frac{\partial m}{g_2 \partial \xi_2} \right\},$$

$$(5.10) \quad \frac{\partial n_2}{\partial t} = -\frac{1}{\sin \chi} \left\{ (u_2 - v_2 \cos \chi) \frac{\partial m}{g_1 \partial \xi_1} + (v_2 - u_2 \cos \chi) \frac{\partial m}{g_2 \partial \xi_2} \right\}.$$

These are precisely the same equations for  $n_1$  and  $n_2$  if we substitute the expression (5.6) for  $\left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$  in the second part of the ray equations, i.e., the equations (2.4) for  $n_1$  and  $n_2$ . Thus, we have derived (2.4) from the differential form of KCL.

## 6. ENERGY TRANSPORT EQUATION FROM A WNLRT FOR A POLYTROPIC GAS AND THE COMPLETE SET OF EQUATIONS

In this section we shall derive a closure relation in a conservation form for the 3D-KCL so that we get a completely determined system of conservation laws. Let the mass density, fluid velocity and gas pressure in a polytropic gas [7] be denoted

by  $\varrho$ ,  $\mathbf{q}$  and  $p$ . Consider now a high frequency small amplitude curved wavefront  $\Omega_t$  running into a polytropic gas in a uniform state and at rest ( $\varrho_0 = \text{constant}$ ,  $\mathbf{q} = 0$  and  $p_0 = \text{constant}$ , [13]-section 6.1). Then a perturbation in the state of the gas on  $\Omega_t$  can be expressed in terms of an amplitude  $w$  and is given by

$$(6.1) \quad \varrho - \varrho_0 = \left(\frac{\varrho_0}{a_0}\right) w, \quad \mathbf{q} = \mathbf{n}w, \quad p - p_0 = \varrho_0 a_0 w.$$

where  $a_0$  is the sound velocity in the undisturbed medium  $= \sqrt{\gamma p_0 / \varrho_0}$  and  $w$  is a quantity of small order, say  $\mathcal{O}(\epsilon)$ . Let us remind, what we stated in the section 1, all dependent variables are dimensional in this (and only in this) paragraph. Note that  $w$  here has the dimension of velocity.

The amplitude  $w$  is related to the non-dimensional normal velocity  $m$  of  $\Omega_t$  by

$$(6.2) \quad m = 1 + \frac{\gamma + 1}{2} \frac{w}{a_0}.$$

The operator  $\frac{d}{dt} = \frac{\partial}{\partial t} + m\langle \mathbf{n}, \nabla \rangle$  in space-time becomes simply the partial derivative  $\frac{\partial}{\partial t}$  in the ray coordinate system  $(\xi_1, \xi_2, t)$ . Hence the energy transport equation of the WNLRT ([13]-equation (6.1.3)) in non-dimensional coordinates becomes

$$(6.3) \quad m_t = (m - 1)\Omega = -\frac{1}{2}(m - 1)\langle \nabla, \mathbf{n} \rangle,$$

where the italic symbol  $\Omega$  is the mean curvature of the wavefront  $\Omega_t$ . Ray tube area  $A$  for any ray system ([18]-pages 244, 280, [13]-relation (2.2.23)) is related to the mean curvature  $\Omega$  (we write here in non-dimensional variables) by

$$(6.4) \quad \frac{1}{A} \frac{\partial A}{\partial l} = -2\Omega, \quad \frac{\partial}{\partial l} \text{ in ray coordinates,}$$

where  $l$  is the arc length along a ray. In non-dimensional variables  $dl = m dt$ . From (6.3) and (6.4) we get

$$(6.5) \quad \frac{2m_t}{m - 1} = -\frac{1}{mA} A_t.$$

This leads to a conservation law, which we accept to be the required one,

$$(6.6) \quad \left\{ (m - 1)^2 e^{2(m-1)} A \right\}_t = 0.$$

Integration gives  $(m - 1)^2 e^{2(m-1)} A = F(\xi_1, \xi_2)$ , where  $F$  is an arbitrary function of  $\xi_1$  and  $\xi_2$ . The ray tube area  $A$  is given by  $A = g_1 g_2 \sin \chi$ , where  $\chi$  is defined by (3.4). In order that  $A$  is positive, we need to choose  $0 < \chi < \pi$ . Now the energy conservation equation becomes

$$(6.7) \quad \left\{ (m - 1)^2 e^{2(m-1)} g_1 g_2 \sin \chi \right\}_t = 0.$$

After a few steps of calculation (see Appendix D), the differential form of this conservation law becomes

$$(6.8) \quad g_2 g_{1t} + g_1 g_{2t} + g_1 g_2 \cot \chi \left\{ -\frac{n_2}{u_3} u_{1t} + \frac{n_1}{u_3} u_{2t} + \frac{n_2}{v_3} v_{1t} - \frac{n_1}{v_3} v_{2t} \right\} + \frac{2g_1 g_2 m}{m - 1} m_t = 0$$

or

$$(6.9) \quad a_{51} u_{1t} + a_{52} u_{2t} + a_{53} v_{1t} + a_{54} v_{2t} + a_{55} m_t + a_{56} g_{1t} + a_{57} g_{2t} = 0, \text{ say.}$$

**The complete set of conservation laws for the weakly nonlinear ray theory (WNLRT) for a polytropic gas are:** the six equations in (3.5)-(3.6) and the equation (6.7). The equations (3.7) need to be satisfied at any fixed  $t$ , say at  $t = 0$ . A complete set of equations of WNLRT in differential form are: the equation (4.15)-(4.18), (6.8) and the two equations (4.12) and (4.13), i.e.,

$$(6.10) \quad g_{1t} + b_{61}^{(1)} u_{1\xi_1} + b_{62}^{(1)} u_{2\xi_1} = 0,$$

$$(6.11) \quad g_{2t} + b_{73}^{(2)} v_{1\xi_2} + b_{74}^{(2)} v_{2\xi_2} = 0,$$

where the coefficients are given in Appendix B.

A matrix form of these equations for the vector  $\mathbf{U} = (u_1, u_2, v_1, v_2, m, g_1, g_2)^T$  is

$$(6.12) \quad A\mathbf{U}_t + B^{(1)}\mathbf{U}_{\xi_1} + B^{(2)}\mathbf{U}_{\xi_2} = 0,$$

where

$$(6.13) \quad A = \begin{bmatrix} g_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_2 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(6.14) \quad B^{(1)} = \begin{bmatrix} b_{11}^{(1)} & b_{12}^{(1)} & b_{13}^{(1)} & b_{14}^{(1)} & -n_1 & 0 & 0 \\ b_{21}^{(1)} & b_{22}^{(1)} & b_{23}^{(1)} & b_{24}^{(1)} & -n_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{61}^{(1)} & b_{62}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$(6.15) \quad B^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{31}^{(2)} & b_{32}^{(2)} & b_{33}^{(2)} & b_{34}^{(2)} & -n_1 & 0 & 0 \\ b_{41}^{(2)} & b_{42}^{(2)} & b_{43}^{(2)} & b_{44}^{(2)} & -n_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{73}^{(2)} & b_{74}^{(2)} & 0 & 0 & 0 \end{bmatrix}.$$

## 7. EIGENVALUES AND EIGENVECTORS OF THE EQUATIONS OF WNLRT IN TERMS THE UNKNOWNNS $(n_1, n_2, m, g_1, g_2)$

Let us define two operators

$$(7.1) \quad \frac{\partial}{\partial \lambda_1} = n_1 \frac{\partial}{\partial x_3} - n_3 \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial \lambda_2} = n_3 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_3}.$$

They represent derivatives in two independent tangential directions on  $\Omega_t$  and hence the operator  $\mathbf{L}$  in (2.4) can be expressed in terms of  $\frac{\partial}{\partial \lambda_1}$  and  $\frac{\partial}{\partial \lambda_2}$ . Two independent

equations in (2.4), say for  $n_1$  and  $n_2$ , can be written as

$$(7.2) \quad \frac{\partial n_1}{\partial t} - \frac{n_2^2 + n_3^2}{n_3} \frac{\partial m}{\partial \lambda_1} - \frac{n_1 n_2}{n_3} \frac{\partial m}{\partial \lambda_2} = 0,$$

$$(7.3) \quad \frac{\partial n_2}{\partial t} + \frac{n_1 n_2}{n_3} \frac{\partial m}{\partial \lambda_1} + \frac{n_1^2 + n_3^2}{n_3} \frac{\partial m}{\partial \lambda_2} = 0.$$

The expression  $\langle \nabla, \mathbf{n} \rangle$ , when we use  $n_3^2 = 1 - n_1^2 - n_2^2$ , can also be written in terms of the operators  $\frac{\partial}{\partial \lambda_1}$  and  $\frac{\partial}{\partial \lambda_2}$ . The transport equation (6.3) now takes the form

$$(7.4) \quad \frac{\partial m}{\partial t} - \frac{(m-1)}{2n_3} \frac{\partial n_1}{\partial \lambda_1} + \frac{(m-1)}{2n_3} \frac{\partial n_2}{\partial \lambda_2} = 0.$$

The equations of WNLRT in terms of unknowns  $n_1, n_2$  and  $m$  are just the three equations (7.2)-(7.4). However, we have shown in section 5 that (7.2) and (7.3) along with (2.3) form a system equivalent to the KCL. Thus the system of equations (7.2)-(7.4) is equivalent to a bigger system of seven equations (6.12) in terms of another set of variables  $\{u_1, u_2, v_1, v_2, m, g_1, g_2\}$ .

For some analysis later on, it is worth adding to the equations (7.2)-(7.4) equations for  $g_1$  and  $g_2$  also in terms of  $\frac{\partial}{\partial \lambda_1}$  and  $\frac{\partial}{\partial \lambda_2}$ . But this can be done only by freezing the operators and the derivatives at a given point.

**Freezing of coefficients at a given point  $P_0(\mathbf{x}_0)$  of  $\Omega_t$**

In the definition (7.1),  $\lambda_1$  and  $\lambda_2$  are not variables, but  $\frac{\partial}{\partial \lambda_1}$  and  $\frac{\partial}{\partial \lambda_2}$  are simply symbols for the two operators. It is quite unlikely that we can globally choose  $\mathbf{u}$  to be  $\frac{1}{\sqrt{n_1^2 + n_3^2}}(-n_3, 0, n_1)$  and  $\mathbf{v}$  to be  $\frac{1}{\sqrt{n_2^2 + n_3^2}}(0, n_3, -n_2)$ . However, the result mentioned at the end of the section 3 tells us that a given point  $P_0(\mathbf{x}_0)$ , we can choose

$$(7.5) \quad \mathbf{u}_0 = \frac{1}{\sqrt{n_1^2 + n_3^2}}(-n_3, 0, n_1), \quad \mathbf{v}_0 = \frac{1}{\sqrt{n_2^2 + n_3^2}}(0, n_3, -n_2).$$

Similarly at this point we can choose

$$(7.6) \quad g_1 = g_{10} \text{ and } g_2 = g_{20}.$$

where  $g_{10}$  and  $g_{20}$  are arbitrary values. Now at  $P_0$

$$(7.7) \quad \begin{aligned} \frac{1}{g_{10}} \frac{\partial}{\partial \xi_1} &= \text{space rate of change in the direction of } \mathbf{u}_0 \\ &= \langle \mathbf{u}_0, \nabla \rangle = \frac{1}{\sqrt{n_1^2 + n_3^2}} \frac{\partial}{\partial \lambda_1}. \end{aligned}$$

Similarly at  $P_0$

$$(7.8) \quad \frac{1}{g_{20}} \frac{\partial}{\partial \xi_2} = \frac{1}{\sqrt{n_2^2 + n_3^2}} \frac{\partial}{\partial \lambda_2}.$$

Freezing the equations (7.2), (7.3) and (7.4) at  $P_0$  and using (7.7) and (7.8) we get at  $P_0$

$$(7.9) \quad \frac{\partial n_1}{\partial t} - \frac{(n_2^2 + n_3^2)\sqrt{n_1^2 + n_3^2}}{n_3 g_{10}} \frac{\partial m}{\partial \xi_1} - \frac{n_1 n_2 \sqrt{n_2^2 + n_3^2}}{n_3 g_{20}} \frac{\partial m}{\partial \xi_2} = 0,$$

$$(7.10) \quad \frac{\partial n_2}{\partial t} + \frac{n_1 n_2 \sqrt{n_1^2 + n_3^2}}{n_3 g_{10}} \frac{\partial m}{\partial \xi_1} + \frac{(n_1^2 + n_3^2)\sqrt{n_2^2 + n_3^2}}{n_3 g_{20}} \frac{\partial m}{\partial \xi_2} = 0,$$

$$(7.11) \quad \frac{\partial m}{\partial t} - \frac{(m-1)\sqrt{n_1^2 + n_3^2}}{2n_3 g_{10}} \frac{\partial n_1}{\partial \xi_1} + \frac{(m-1)\sqrt{n_2^2 + n_3^2}}{2n_3 g_{20}} \frac{\partial n_2}{\partial \xi_2} = 0.$$

The eigenvalues of these three frozen equations are

$$(7.12) \quad \mu_{1,2} = \pm \left[ \frac{m-1}{2n_3^2} \left\{ (n_2^2 + n_3^2) \bar{e}_1^2 + 2n_1 n_2 \bar{e}_1 \bar{e}_2 + (n_1^2 + n_3^2) e_2^{-2} \right\} \right]^{1/2}, \quad \mu_3 = 0,$$

where

$$(7.13) \quad \bar{e}_1 = \frac{\sqrt{n_1^2 + n_3^2}}{g_{10}} e_1, \quad \bar{e}_2 = \frac{\sqrt{n_2^2 + n_3^2}}{g_{20}} e_2$$

and  $(e_1, e_2)$  is an arbitrary nonzero 2-D vector. Since these are distinct, the eigenspace is complete. It is easy to see that

$$(7.14) \quad (n_2^2 + n_3^2) \bar{e}_1^2 + 2n_1 n_2 \bar{e}_1 \bar{e}_2 + (n_1^2 + n_3^2) \bar{e}_2^2 = (n_2 \bar{e}_1 + n_1 \bar{e}_2)^2 + n_3^2 (\bar{e}_1^2 + \bar{e}_2^2) > 0.$$

Thus the frozen system of 3 equations (7.9)-(7.11) is hyperbolic if  $m > 1$  and has elliptic nature if  $m < 1$  (not strictly elliptic because one eigenvalue is real).

To the equations (7.9)-(7.11), we can add the equations for  $g_1$  and  $g_2$  in terms of the variables  $n_1$  and  $n_2$ . Writing differential form of (3.5) and taking inner product with  $\mathbf{u}$ , we get  $g_{1t} = m \langle \mathbf{u}, \mathbf{n}_{\xi_1} \rangle$ . We eliminate the derivative of  $n_3$  by using  $n_3^2 = 1 - n_1^2 - n_2^2$  and get

$$(7.15) \quad g_{1t} = \frac{m}{n_3} \left\{ (n_3 u_1 - n_1 u_3) n_{1\xi_1} - (n_2 u_3 - n_3 u_2) n_{2\xi_1} \right\}.$$

We freeze this equation at  $P_0$  and use (7.5), then the equation becomes

$$(7.16) \quad g_{1t} + \frac{m\sqrt{n_1^2 + n_3^2}}{n_3} n_{1\xi_1} + \frac{mn_1 n_2}{n_3 \sqrt{n_1^2 + n_3^2}} n_{2\xi_1} = 0.$$

Similarly, we can get the frozen equation for  $g_2$  at  $P_0$  as

$$(7.17) \quad g_{2t} - \frac{mn_1 n_2}{n_3 \sqrt{n_2^2 + n_3^2}} n_{1\xi_2} - \frac{m\sqrt{n_2^2 + n_3^2}}{n_3} n_{2\xi_2} = 0.$$

It is important to note that these two equations (like the equations (7.9)-(7.11)) are frozen equations at  $P_0$  with special directions  $\mathbf{u}_0$  and  $\mathbf{v}_0$ .

Since  $g_{10}$  and  $g_{20}$  appear in (7.9)-(7.11), we should actually consider the system of five frozen equation (7.9)-(7.11), (7.16) and (7.17). The eigenvalues of this system are

$$(7.18) \quad \mu_1, \mu_2, \mu_3 = 0, \quad \mu_4 = 0, \quad \mu_5 = 0,$$

where  $\mu_1$  and  $\mu_2$  are given by (7.12). It is now simple to check that the number of linearly independent eigenvectors corresponding to the triple eigenvalue 0 is only 2. The system now becomes degenerate. This is an important result which will be noticed to be true for the system of 7 equations for the vector  $(u_1, u_2, v_1, v_2, m, g_1, g_2)$ .

8. EIGENVALUES AND EIGENVECTORS OF WNLRT IN TERMS OF  
( $u_1, u_2, v_1, v_2, m, g_1, g_2$ )

We have not been able to find out expressions for the eigenvalues of the system of equations (6.12) directly by solving the 7th degree equation for the eigenvalues. We can find them in a special case by choosing the vectors  $\mathbf{u} = \mathbf{u}'$  and  $\mathbf{v} = \mathbf{v}'$ , where  $\mathbf{u}'$  and  $\mathbf{v}'$  are orthogonal at  $P_0$  and then freezing the coefficients at this point.

**8.1. Freezing of the coefficients at  $P_0$  where  $\mathbf{u}'$  and  $\mathbf{v}'$  are orthogonal.**

When  $\mathbf{u}'$  is orthogonal to  $\mathbf{v}'$ ,  $\cos \chi' = 0$  and  $\sin \chi' = 1$ . The entries in the coefficient matrices  $A', B'^{(1)}$  and  $B'^{(2)}$  of the system (6.12) simplify considerably. Then the equation for the eigenvalues, i.e.,  $\det(-\nu A' + e'_1 B'^{(1)} + e'_2 B'^{(2)}) = 0$ , where  $(e'_1, e'_2) \in \mathbb{R}^2 \setminus (0, 0)$ , becomes

$$(8.1) \quad \det \begin{bmatrix} -\nu g'_1 & 0 & \frac{mu'_2 v'_1}{v'_3} e'_1 & \frac{mu'_1 v'_1}{v'_3} e'_1 & -n_1 e'_1 & 0 & 0 \\ 0 & -\nu g'_1 & \frac{mu'_2 v'_2}{v'_3} e'_1 & \frac{mu'_1 v'_2}{v'_3} & -n_2 e'_1 & 0 & 0 \\ -\frac{mu'_1 v'_2}{u'_3} e'_2 & \frac{mu'_1 v'_1}{u'_3} e'_2 & -\nu g'_2 & 0 & -n_1 e'_2 & 0 & 0 \\ -\frac{mu'_2 v'_2}{u'_3} e'_2 & \frac{mu'_2 v'_1}{u'_3} e'_2 & 0 & -\nu g'_2 & -n_2 e'_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu \frac{2m}{m-1} g'_1 g'_2 & -\nu g'_2 & -\nu g'_1 \\ -\frac{mv'_2}{u'_3} e'_1 & \frac{mv'_1}{u'_3} e'_1 & 0 & 0 & 0 & -\nu & 0 \\ 0 & 0 & \frac{mu'_2}{v'_3} e'_2 & -\frac{mu'_1}{v'_3} e'_2 & 0 & 0 & -\nu \end{bmatrix} = 0.$$

A long calculation leads to the following eigenvalues

$$(8.2) \quad \nu_{1,2} = \pm \left\{ \frac{(m-1)(e_1'^2 g_2' + e_2'^2 g_1')}{2g_1'^2 g_2'^2} \right\}^{1/2}, \quad \nu_3 = \nu_4 = \nu_5 = \nu_6 = \nu_7 = 0.$$

It is also found that the number of independent eigenvectors corresponding to the multiple eigenvalue 0 is 4 resulting in the loss of hyperbolicity of the system for  $m > 1$ .

**8.2. Numerical computation of the eigenvalues and eigenvectors for the general case.**

In this case we take any point  $P$  on  $\Omega_t$  and choose  $g_1 = 1, g_2 = 1$ . Now we choose different values of  $m > 1$  and  $< 1$  and of vectors  $\mathbf{u}$  and  $\mathbf{v}$  such that  $u_3 \neq 0$  and  $v_3 \neq 0$ . Choice of  $u_3 \neq 0$  and  $v_3 \neq 0$  is required because we discounted the equations for  $u_3$  and  $v_3$  in the section 4 and this lead to appearance of  $u_3$  and  $v_3$  in the denominators of many coefficients in the equations (6.12). We solved equation  $\det(-\lambda A + e_1 B^{(1)} + e_2 B^{(2)}) = 0$  numerically for computing the eigenvalues  $\lambda_i, i = 1, 2, \dots, 7$ , for a number of values of scalars  $e_1$  and  $e_2$ . To simplify the numerical computation we used the relations (4.21) and (4.23). In computation of the results presented in Table 1 and Table 2 we have taken  $e_1 = e_2 = 1/\sqrt{2}$ .

All results for different choices of  $\mathbf{u}$  and  $\mathbf{v}$  gave values of  $\lambda_3 = \dots = \lambda_7 = 0$  and  $\lambda_1 (= -\lambda_2)$  real for  $m > 1$  and purely imaginary for  $m < 1$ . The values of  $\lambda_1$  can be seen in the table for some choices of  $\mathbf{u}$  and  $\mathbf{v}$ . From numerical experiment, we postulate that *the  $7 \times 7$  system (6.12) has two distinct eigenvalues  $\lambda_1$  and  $\lambda_2 = -\lambda_1$  for  $m \neq 1$  and five coincident eigenvalues  $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$ . The eigenvalues  $\lambda_1$  and  $\lambda_2$  are real for  $m > 1$ , imaginary for  $m < 1$  and corresponding*

TABLE 1. for  $m > 1$ ,  $\lambda_1$  and  $\lambda_2$  are real

Sl.No.	$m$	$\mathbf{u}$ (3-components)	$\mathbf{v}$ (3-components)	$\lambda_1(\lambda_2 = -\lambda_1)$
1	1.5	(0.3093, 0.8385, 0.4486)	(0.5681, 0.3704, 0.7349)	0.3492
2	1.5	(0.5466, 0.4449, 0.7095)	(0.6946, 0.6213, 0.3627)	0.8204
3	1.5	(0.8801, 0.1730, 0.4421)	(0.2523, 0.8757, 0.4116)	0.3145
4	1.5	(0.1365, 0.0118, 0.9906)	(0.8939, 0.1991, 0.4016)	0.3662
5	1.5	(0.6614, 0.2844, 0.6940)	(0.4692, 0.0648, 0.8807)	0.8559
6	1.5	(0.5828, 0.4235, 0.6935)	(0.5155, 0.3340, 0.7891)	2.4591
7	1.5	(0.2260, 0.5798, 0.7828)	(0.7604, 0.5298, 0.3757)	0.2122
8	1.5	(0.2091, 0.3798, 0.9011)	(0.4611, 0.5678, 0.6819)	0.8882
9	1.5	(0.0592, 0.6029, 0.7956)	(0.0503, 0.4154, 0.9083)	1.6031
10	1.5	(0.8744, 0.0150, 0.4850)	(0.4387, 0.4983, 0.7478)	0.3412

TABLE 2. for  $m < 1$ ,  $\lambda_1$  and  $\lambda_2$  are imaginary

Sl.No.	$m$	$\mathbf{u}$	$\mathbf{v}$	$\lambda_1(\lambda_2 = -\lambda_1)$
1	0.375	(0.9501, 0.2311, 0.2094)	(0.6068, 0.4860, 0.6289)	1.3135 <i>i</i>
2	0.375	(0.7621, 0.4565, 0.4592)	(0.0185, 0.8214, 0.5700)	0.8764 <i>i</i>
3	0.375	(0.1763, 0.4057, 0.8968)	(0.4103, 0.8937, 0.1818)	0.5413 <i>i</i>
4	0.375	(0.3529, 0.8132, 0.4629)	(0.0099, 0.1389, 0.9903)	0.5627 <i>i</i>
5	0.375	(0.1987, 0.6038, 0.7720)	(0.2722, 0.1988, 0.9415)	1.1723 <i>i</i>
6	0.375	(0.7468, 0.4451, 0.4942)	(0.4186, 0.8462, 0.3296)	0.9160 <i>i</i>
7	0.375	(0.2026, 0.6721, 0.7122)	(0.8381, 0.0196, 0.5451)	0.5123 <i>i</i>
8	0.375	(0.3795, 0.8318, 0.4051)	(0.5028, 0.7095, 0.4938)	3.8255 <i>i</i>
9	0.375	(0.3046, 0.1897, 0.9334)	(0.1934, 0.6822, 0.7051)	1.2948 <i>i</i>
10	0.375	(0.5417, 0.1509, 0.8269)	(0.6979, 0.3784, 0.6081)	2.2349 <i>i</i>

to the multiple eigenvalue 0 of multiplicity 5, there exist only 4 linearly independent eigenvectors. We shall prove this postulate in the next subsection.

**8.3. Transformation of frozen coordinates at a given point to get the eigenvalues in the most general case.** We have not been able to get the expressions for the eigenvalues of the system (6.11) in its general form by solving the algebraic equation  $\det(-\lambda A + e_1 B^{(1)} + e_2 B^{(2)}) = 0$ . We could find these expressions in a particular case when  $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')$ , with  $\langle \mathbf{u}', \mathbf{v}' \rangle = 0$ . In this subsection, we shall obtain the expressions of the nonzero eigenvalues in the general case from the expressions of  $\nu_1$  and  $\nu_2$  in (8.2) by transforming the set  $\{\mathbf{u}', \mathbf{v}'\}$  to  $\{\mathbf{u}, \mathbf{v}\}$  in the tangent plane at a given point  $P_0$  on  $\Omega_t$  (all vectors frozen at the point  $P_0$ ).

Let  $(\eta_1, \eta_2)$  be a coordinate system on  $\Omega_t$ , which is orthogonal at  $P_0$  with unit tangent vectors  $\mathbf{u}'$  and  $\mathbf{v}'$  to the curves  $\eta_2 = \text{constant}$  and  $\eta_1 = \text{constant}$  respectively at this point. Then

$$(8.3) \quad \mathbf{u}' = \frac{1}{g'_1} \frac{\partial \mathbf{x}}{\partial \eta_1}, \quad \mathbf{v}' = \frac{1}{g'_2} \frac{\partial \mathbf{x}}{\partial \eta_2}.$$

Let the set  $\{\mathbf{u}', \mathbf{v}'\}$  and  $\{\mathbf{u}, \mathbf{v}\}$  at  $P_0$  be related by

$$(8.4) \quad \mathbf{u}' = \gamma_1 \mathbf{u} + \delta_1 \mathbf{v}, \quad \mathbf{v}' = \gamma_2 \mathbf{u} + \delta_2 \mathbf{v}.$$

From expressions (8.3) for  $\mathbf{u}', \mathbf{v}'$ , similar expressions for  $\mathbf{u}, \mathbf{v}$  in (5.4) and relations (8.4) we get the following transformation of derivations

$$(8.5) \quad \frac{1}{g'_1} \frac{\partial}{\partial \eta_1} = \gamma_1 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_1 \frac{1}{g_2} \frac{\partial}{\partial \xi_2},$$

$$(8.6) \quad \frac{1}{g'_2} \frac{\partial}{\partial \eta_2} = \gamma_2 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_2 \frac{1}{g_2} \frac{\partial}{\partial \xi_2}.$$

The frozen system of equations for  $\mathbf{U}' = (u'_1, u'_2, v'_1, v'_2, m, g'_1, g'_2)$  in terms of operators  $\frac{\partial}{\partial \eta_1}$  and  $\frac{\partial}{\partial \eta_2}$ , i.e., the equation

$$(8.7) \quad A \frac{\partial \mathbf{U}'}{\partial t} + B'^{(1)} \frac{\partial \mathbf{U}'}{\partial \eta_1} + B'^{(2)} \frac{\partial \mathbf{U}'}{\partial \eta_2} = 0.$$

gets transformed into the system

$$(8.8) \quad A \frac{\partial \mathbf{U}}{\partial t} + g'_1 B'^{(1)} \left( \gamma_1 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_1 \frac{1}{g_2} \frac{\partial}{\partial \xi_2} \right) \mathbf{U} \\ + g'_2 B'^{(2)} \left( \gamma_2 \frac{1}{g_1} \frac{\partial}{\partial \xi_1} + \delta_2 \frac{1}{g_2} \frac{\partial}{\partial \xi_2} \right) \mathbf{U} = 0.$$

Comparing this equation with (6.12) we get

$$(8.9) \quad g_1 B^{(1)} = \gamma_1 g'_1 B'^{(1)} + \gamma_2 g'_2 B'^{(2)},$$

$$(8.10) \quad g_2 B^{(2)} = \delta_1 g'_1 B'^{(1)} + \delta_2 g'_2 B'^{(2)}.$$

The characteristic equation of (6.12) is

$$(8.11) \quad \det \left( -\lambda A + e_1 B^{(1)} + e_2 B^{(2)} \right) = 0$$

which with the help of (8.9) and (8.10) becomes

$$(8.12) \quad \det \left[ -\lambda A + g'_1 \left( \frac{e_1}{g_1} \gamma_1 + \frac{e_2}{g_2} \delta_1 \right) B'^{(1)} \right. \\ \left. + g'_2 \left( \frac{e_1}{g_1} \gamma_2 + \frac{e_2}{g_2} \delta_2 \right) B'^{(2)} \right] = 0.$$

This is same as the characteristic equation of (8.7) with an eigenvalue  $\lambda'$  if

$$(8.13) \quad \lambda' = \lambda, \quad \frac{e'_1}{g'_1} = \gamma_1 \frac{e_1}{g_1} + \delta_1 \frac{e_2}{g_2}, \quad \frac{e'_2}{g'_2} = \gamma_2 \frac{e_1}{g_1} + \delta_2 \frac{e_2}{g_2}$$

Thus, we have proved an important result

**Theorem 8.1.** *Let  $\lambda'$  be an expression of an eigenvalue of (8.7) in terms of  $e'_1/g'_1$  and  $e'_2/g'_2$ . Then the expression for the same eigenvalue of (6.12) in terms of  $e_1/g_1$  and  $e_2/g_2$  can be obtained from it by replacing  $e'_1/g'_1$  and  $e'_2/g'_2$  by (8.13).*

Using this theorem, we derive the expressions for the eigenvalues of (6.12) from those of (8.2). The first result that we conclude is that these eigenvalues are

$$(8.14) \quad \lambda_1 \neq 0, \quad \lambda_2 (= -\lambda_1), \quad \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$$

and then from (8.2) we get the expression for  $\lambda_1$  as

$$(8.15) \quad \lambda_1 = \left[ \frac{m-1}{2} \left\{ (\gamma_1^2 + \gamma_2^2) \frac{e_1^2}{g_1^2} + 2(\gamma_1\delta_1 + \gamma_2\delta_2) \frac{e_1}{g_1} \frac{e_2}{g_2} + (\delta_1^2 + \delta_2^2) \frac{e_2^2}{g_2^2} \right\} \right]^{1/2}.$$

The rank of the pencil matrix for the eigenvalue  $\lambda = 0$ , i.e., the rank of  $e_1B^{(1)} + e_2B^{(2)}$  will be the same as the rank of  $e_1'B^{(1)} + e_2'B^{(2)}$  when the relations (8.9) and (8.10) are valid and hence it would be 3. Thus, the number of linearly independent eigenvectors corresponding to  $\lambda = 0$  is only 4. We have now proved the main theorem.

**Theorem 8.2.** *The system (6.12) has 7 eigenvalues  $\lambda_1, \lambda_2 (= -\lambda_1), \lambda_3 = \lambda_4 = \dots = \lambda_7 = 0$ , where  $\lambda_1$  and  $\lambda_2$  are real for  $m > 1$  and purely imaginary for  $m < 1$ . Further, the dimension of the eigenspace corresponding to the multiple eigenvalue 0 is 4.*

**8.4. Application of the theory in the section 8.3.** We have shown the equivalence of the ray equations to the differential form of KCL in section 5. Hence, we expect to derive the expressions of the eigenvalues  $\mu_1$  and  $\mu_2$  of the section 7 from the eigenvalues  $\nu_1$  and  $\nu_2$  simply by using the general formula (8.15). It is really remarkable to do this derivation and show the power of the transformation of coordinates introduced in the last subsection.

Note that the frozen coordinates  $(\xi_1, \xi_2, t)$  at  $P_0$  are associated with the unit vectors  $\mathbf{u}_0$  and  $\mathbf{v}_0$  given in (7.5). We now choose orthogonal coordinates  $(\eta_1, \eta_2)$  at  $P_0$  such that these coordinates are associated with unit tangent vectors

$$(8.16) \quad \mathbf{u}'_0 = \mathbf{u}_0 = \frac{1}{\sqrt{n_1^2 + n_3^2}}(-n_3, 0, n_1)$$

and

$$(8.17) \quad \mathbf{v}'_0 = \frac{1}{\sqrt{n_1^2 + n_3^2}}(-n_1n_2, n_1^2 + n_3^2, -n_2n_3).$$

Note that  $\mathbf{v}'_0$  is not only orthogonal to  $\mathbf{u}'_0$  but also to  $\mathbf{n}$ . Now, the relation (8.4) becomes

$$(8.18) \quad \mathbf{u}'_0 = \gamma_1\mathbf{u}_0 + \delta_1\mathbf{v}_0, \quad \mathbf{v}'_0 = \gamma_2\mathbf{u}_0 + \delta_2\mathbf{v}_0,$$

where

$$(8.19) \quad \gamma_1 = 1, \quad \delta_1 = 0; \quad \gamma_2 = \frac{n_1n_2}{n_3}, \quad \delta_2 = \frac{1}{n_3}\sqrt{n_1^2 + n_3^2}\sqrt{n_2^2 + n_3^2}.$$

Substituting these values in the general formula (8.15) we find that the eigenvalue  $\nu_1$  (in (8.2)) of the orthogonal system becomes  $\mu_1$ , where

$$(8.20) \quad \mu_1^2 = \frac{m-1}{2n_3^2} \left[ (n_3^2 + n_1^2n_2^2) \frac{e_1^2}{g_{10}^2} + (n_1^2 + n_3^2)(n_2^2 + n_3^2) \frac{e_2^2}{g_{20}^2} + 2n_1n_2\sqrt{n_1^2 + n_3^2}\sqrt{n_2^2 + n_3^2} \frac{e_1}{g_{10}} \frac{e_2}{g_{20}} \right].$$

We note that

$$(8.21) \quad n_3^2 + n_1^2n_2^2 = n_3^2(n_1^2 + n_2^2 + n_3^2) + n_1^2n_2^2 = (n_1^2 + n_3^2)(n_2^2 + n_3^2).$$

Using this result in (8.21), we find that  $\mu_1$  and  $\mu_2 (= -\mu_1)$  here are exactly the same as  $\mu_1$  and  $\mu_2$  in (7.12).

Thus we have obtained a beautiful result-*derivation of the eigenvalues of the a system (say the system-I) consisting of ray equations and the energy transport equation from the differential form of another system (say, system II) consisting of the KCL along with the energy transport equation.* The process can be reversed. We can obtain the eigenvalues of the system-II from those of the smaller and simpler system-I. The invariance of eigenvalues and eigenvectors of two equivalence systems of same number equations is well known (see [15]-Theorem 6.1, page 220) but the invariance which we see here is in two system of different number of equations.

The problem of evolution of a moving surface  $\Omega_t$  in  $\mathbb{R}^3$  is quite complex. Ray theory represented by the system I is a complete system of equations except that it would not describe formation and propagation of kink lines. This system in ray-coordinates has 3 eigenvalues  $\mu_1, \mu_2$  and  $\mu_3$  given in (7.12). The first two non zero eigenvalues for  $m > 1$  carry with them changes in the geometry of  $\Omega_t$  and the amplitude  $w$  (or  $m$ ) on  $\Omega_t$ . The third zero eigenvalue carries with it the total energy and represents conservation of the total energy in a ray tube. It will be interesting to study in  $(\xi_1, \xi_2, t)$ -space geometry of bicharacteristics associated with these characteristic fields. The system II coming from KCL is essential to study the evolution of  $\Omega_t$  with singularities. However, this leads to 4 more eigenvalues, which are all equal to  $\lambda_3 = 0$ . These additional eigenvalues are results of an increase in the dependent variables  $u_1, u_2, v_1, v_2, m, g_1, g_2$  which are necessary in the formulation of KCL (in reality only three variables  $n_1, n_2, m$  along with the equations (2.3) for  $\mathbf{x}$  suffice to describe the evolution of  $\Omega_t$ ). We need to study some exact and numerical solutions of the KCL with energy transport equation to see the effect of these additional eigenvalues, which cause a loss in the hyperbolicity of the system of  $m > 1$  (the number of independent eigenvectors corresponding to  $\lambda_i = 0, i = 3, 4, \dots, 7$ , is only four).

## 9. FORMULATION OF THE RAY COORDINATES FOR A PARTICULAR SURFACE

Let the initial position of a weakly nonlinear wavefront  $\Omega_t$  be given as

$$(9.1) \quad \Omega_0: x_3 = f(x_1, x_2).$$

On  $\Omega_0$ , we choose

$$(9.2) \quad \xi_1 = x_1, \quad \xi_2 = x_2$$

then

$$(9.3) \quad \Omega_0: x_{10} = \xi_1, \quad x_{20} = \xi_2, \quad x_{30} = f(\xi_1, \xi_2)$$

and

$$(9.4) \quad g_{10} = \sqrt{1 + f_{\xi_1}^2}, \quad \mathbf{u}_0 = \frac{(1, 0, f_{\xi_1})}{\sqrt{1 + f_{\xi_1}^2}},$$

$$(9.5) \quad g_{20} = \sqrt{1 + f_{\xi_2}^2}, \quad \mathbf{v}_0 = \frac{(0, 1, f_{\xi_2})}{\sqrt{1 + f_{\xi_2}^2}}.$$

We can easily check that (3.7) is satisfied on  $\Omega_0$ . The unit normal  $\mathbf{n}_0$  on  $\Omega_0$  is

$$(9.6) \quad \mathbf{n}_0 = -\frac{(f_{\xi_1}, f_{\xi_2}, -1)}{\sqrt{1 + f_{\xi_1}^2 + f_{\xi_2}^2}}$$

in which the sign is so chosen that  $(\mathbf{u}, \mathbf{v}, \mathbf{n})$  form a right handed system. Let the distribution of the front velocity be given by

$$(9.7) \quad m = m_0(\xi_1, \xi_2).$$

We have now completed formulation of the initial data for the KCL (3.5) and (3.6), and the energy transport equation (6.6).

The problem is to find solution of the system (3.5), (3.6) and (6.6) satisfying the initial data given by (9.4), (9.5) and (9.7). Having solved these equations, we can get  $\Omega_t$  by solving the first part of the ray equations (2.3) at least numerically for a number of values of  $\xi_1$  and  $\xi_2$ .

The formulation given in this section has been used by Arun, Lukáčová and Rao [1] to find numerically the evolution of two nonlinear wavefronts axi-symmetric about the  $z$ -axis. The first one is when the initial shape has Gaussian elevation and the front moves outward and the second one is when the initial shape has parabolic depression and the front moves inward. The three dimensional geometrical divergence effect weakens the otherwise curvature effect of a 2-D wavefront in the first case where the mean curvature is negative and reinforces the convergence effect in the second case where the mean curvature is positive. It is very interesting to note that in the second case though the wavefront crosses itself (with a point singularity on it), the solution in  $(\xi_1, \xi_2)$ -space is single valued as expected for a system of conservation laws. They obtain some very interesting results and they also raise some deep mathematical questions regarding existence of  $\delta$ -waves in the solutions, which we plan to study in future.

#### APPENDIX A

**Lemma A.1.** *The third equation in (4.3) can be derived from the first two equations in (4.3)*

*Proof.* The first two equations in (4.3) are

$$(A.1) \quad g_1 u_{1t} = n_1 m_{\xi_1} + m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle u_1 + m n_1 \xi_1,$$

$$(A.2) \quad g_1 u_{2t} = n_2 m_{\xi_1} + m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle u_2 + m n_2 \xi_1.$$

Multiplying (A.1) by  $u_1$  and (A.2) by  $u_2$ , adding the two results, and using  $u_1^2 + u_2^2 = 1 - u_3^2$  and  $u_1 n_1 + u_2 n_2 = -u_3 n_3$ , we get

$$(A.3) \quad -2g_1 u_3 u_{3t} = -2n_3 u_3 m_{\xi_1} - 2m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle u_3^2 - 2m u_3 n_3 \xi_1.$$

Assuming  $u_3 \neq 0$ , we divide this result by  $-2u_3$  and get the third equation in (4.3)  $\square$

#### APPENDIX B

**Calculation of the coefficients  $b_{ij}^{(1)}$ ,  $b_{ij}^{(2)}$**

Substituting (4.7) in (4.3), we get

$$(B.1) \quad \mathbf{u}_t = \frac{1}{g_1} m_{\xi_1} \mathbf{n} + \frac{1}{g_1} m \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle + \frac{m}{g_1 \sin \chi} \left\{ (\mathbf{u} \times \mathbf{v})_{\xi_1} + \frac{\mathbf{n} \langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u} \times \mathbf{v}|} \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1} \right\}.$$

Since  $|\mathbf{u}| = 1$  and  $|\mathbf{v}| = 1$ ,

$$(B.2) \quad u_{3\xi_1} = -\frac{u_1 u_{1\xi_1} + u_2 u_{2\xi_1}}{u_3}, \quad v_{3\xi_1} = -\frac{v_1 v_{1\xi_1} + v_2 v_{2\xi_1}}{v_3}.$$

Hence

$$(B.3) \quad \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle = \frac{1}{u_3} \{ -(n_3 u_1 - n_1 u_3) u_{1\xi_1} + (n_2 u_3 - n_3 u_2) u_{2\xi_1} \}.$$

Substituting the expressions for the components of  $\mathbf{n}$  in terms of those of  $\mathbf{u}$  and  $\mathbf{v}$  from (3.1),

$$(B.4) \quad \begin{aligned} n_3 u_1 - n_1 u_3 &= \frac{1}{\sin \chi} \{ (u_1^2 + u_3^2) v_2 - u_2 (u_1 v_1 + u_3 v_3) \} \\ &= \frac{1}{\sin \chi} \{ (1 - u_2^2) v_2 - u_2 (u_1 v_1 + u_3 v_3) \} \\ &= \frac{1}{\sin \chi} \{ v_2 - u_2 (u_1 v_1 + u_2 v_2 + u_3 v_3) \} \\ &= \frac{1}{\sin \chi} (v_2 - u_2 \cos \chi). \end{aligned}$$

Substituting (B.4) and a similar expression for  $n_2 u_3 - n_3 u_2$  in (B.3), we get

$$(B.5) \quad \langle \mathbf{n}, \mathbf{u}_{\xi_1} \rangle = \frac{1}{u_3 \sin \chi} \{ -(v_2 - u_2 \cos \chi) u_{1\xi_1} + (v_1 - u_1 \cos \chi) u_{2\xi_1} \}.$$

Differentiating the first component of  $\mathbf{u} \times \mathbf{v}$  with respect to  $\xi_1$ , using (B.2) and re-arranging terms

$$(B.6) \quad (\mathbf{u} \times \mathbf{v})_{1\xi_1} = \frac{1}{u_3} \{ u_1 v_2 u_{1\xi_1} + (u_2 v_2 + u_3 v_3) u_{2\xi_1} \} - \frac{1}{v_3} \{ u_2 v_1 v_{1\xi_1} + (u_2 v_2 + u_3 v_3) v_{2\xi_1} \}.$$

Further, using the expressions (4.10) for components of  $\mathbf{n}$  in

$$(B.7) \quad \frac{\langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1}}{|\mathbf{u} \times \mathbf{v}|} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u} \times \mathbf{v}|} \left[ \frac{1}{u_3} \{ (u_3 v_1 - u_1 v_3) u_{1\xi_1} - (u_2 v_3 - u_3 v_2) u_{2\xi_1} \} + \frac{1}{v_3} \{ -(u_3 v_1 - u_1 v_3) v_{1\xi_1} + (u_2 v_3 - u_3 v_2) v_{2\xi_1} \} \right]$$

we get

$$(B.8) \quad \frac{\langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{v} \rangle_{\xi_1}}{|\mathbf{u} \times \mathbf{v}|} = \cos \chi \left( \frac{n_2}{u_3} u_{1\xi_1} - \frac{n_1}{u_3} u_{2\xi_1} - \frac{n_2}{v_3} v_{1\xi_1} + \frac{n_1}{v_3} v_{2\xi_1} \right).$$

We have completed evaluation of all terms in (B.1). Substituting expressions (B.5)-(B.7) in (B.1), we write the equation for  $u_1$  as

$$(B.9) \quad g_1 u_{1t} - n_1 m_\xi + b_{11}^{(1)} u_{1\xi_1} + b_{12}^{(1)} u_{2\xi_1} + b_{13}^{(1)} v_{1\xi_1} + b_{14}^{(1)} v_{2\xi_1} = 0,$$

where the coefficient  $b_{1j}^{(1)}, j = 1, 2, 3, 4$ , are given below. Similarly other coefficients can be evaluated. All these are also recorded below.

**Non-zero elements of the matrices  $A$ ,  $B^{(1)}$  and  $B^{(2)}$**

$$\begin{aligned} a_{11} &= g_1, \quad a_{22} = g_1, \quad a_{33} = g_2, \quad a_{44} = g_2, \quad a_{66} = 1, \quad a_{77} = 1, \\ a_{51} &= -\frac{1}{u_3} g_1 g_2 n_2 \cot \chi, \quad a_{52} = \frac{1}{u_3} g_1 g_2 n_1 \cot \chi, \\ a_{53} &= \frac{1}{v_3} g_1 g_2 n_2 \cot \chi, \quad a_{54} = -\frac{1}{v_3} g_1 g_2 n_1 \cot \chi, \\ a_{55} &= \frac{2m}{m-1} g_1 g_2 a_{56} = g_2, \quad a_{57} = g_1. \end{aligned}$$

$$\begin{aligned}
b_{11}^{(1)} &= -\frac{m}{u_3}(u_1u_2 + n_1n_2) \cot \chi, & b_{12}^{(1)} &= \frac{m}{u_3}(u_1^2 + n_1^2 - 1) \cot \chi, \\
b_{13}^{(1)} &= \frac{m}{v_3 \sin \chi}(u_2v_1 + n_1n_2 \cos \chi), & b_{14}^{(1)} &= -\frac{m}{v_3 \sin \chi}(u_1v_1 + (n_1^2 - 1) \cos \chi), \\
b_{15}^{(1)} &= -n_1.
\end{aligned}$$

$$\begin{aligned}
b_{21}^{(1)} &= -\frac{m}{u_3}(u_2^2 + n_2^2 - 1) \cot \chi, & b_{22}^{(1)} &= \frac{m}{u_3}(u_1u_2 + n_1n_2) \cot \chi, \\
b_{23}^{(1)} &= \frac{m}{v_3 \sin \chi}(u_2v_2 + (n_2^2 - 1) \cos \chi), & b_{24}^{(1)} &= -\frac{m}{v_3 \sin \chi}(u_1v_2 + n_1n_2 \cos \chi), \\
b_{25}^{(1)} &= -n_2.
\end{aligned}$$

$$b_{61}^{(1)} = -\frac{m}{u_3 \sin \chi}(v_2 - u_2 \cos \chi), \quad b_{62}^{(1)} = \frac{m}{u_3 \sin \chi}(v_1 - u_1 \cos \chi).$$

$$\begin{aligned}
b_{31}^{(2)} &= -\frac{m}{u_3 \sin \chi}(u_1v_2 + n_1n_2 \cos \chi), & b_{32}^{(2)} &= \frac{m}{u_3 \sin \chi}(u_1v_1 + (n_1^2 - 1) \cos \chi), \\
b_{33}^{(2)} &= \frac{m}{v_3}(v_1v_2 + n_1n_2) \cot \chi, & b_{34}^{(2)} &= -\frac{m}{v_3}(v_1^2 + n_1^2 - 1) \cot \chi, \\
b_{35}^{(2)} &= -n_1.
\end{aligned}$$

$$\begin{aligned}
b_{41}^{(2)} &= -\frac{m}{u_3 \sin \chi}(u_2v_2 + (n_2^2 - 1) \cos \chi), & b_{42}^{(2)} &= \frac{m}{u_3 \sin \chi}(u_2v_1 + n_1n_2 \cos \chi), \\
b_{43}^{(2)} &= \frac{m}{v_3}(v_2^2 + n_2^2 - 1) \cot \chi, & b_{44}^{(2)} &= -\frac{m}{v_3}(v_1v_2 + n_1n_2) \cot \chi, \\
b_{45}^{(2)} &= -n_2.
\end{aligned}$$

$$b_{73}^{(2)} = \frac{m}{v_3 \sin \chi}(u_2 - v_2 \cos \chi), \quad b_{74}^{(2)} = -\frac{m}{v_3 \sin \chi}(u_1 - v_1 \cos \chi).$$

## APPENDIX C

**Derivation of the equations (5.9) and (5.10)**

We first substitute in (5.7) the expressions for  $u_{1t}, u_{2t}, v_{1t}$  and  $v_{2t}$  from (4.15)-(4.18) and then examine coefficients of  $m_{\xi_i}, u_{1\xi_i}, u_{2\xi_i}, v_{1\xi_i}, v_{2\xi_i}, i = 1, 2$ , in the curly brackets one by one.

$$\begin{aligned}
\text{Coefficient of } \frac{1}{g_1} m_{\xi_1} &= n_1(n_3u_1 - n_1u_3) - n_2(n_2u_3 - n_3u_2) \\
&= (n_1u_1 + n_2u_2)n_3 - (n_1^2 + n_2^2)u_3 \\
&= (-n_3u_3)n_3 - (1 - n_3^2)u_3 \\
&= -u_3.
\end{aligned} \tag{C.1}$$

Similarly, coefficient of  $\frac{1}{g_1}m_{\xi_2} = -v_3$ .

(C.2)

$$\begin{aligned} \text{Coefficient of } \frac{1}{g_1}u_{1\xi_1} &= -b_{11}^{(1)}(n_3u_1 - n_1u_3) + b_{21}^{(1)}(n_2u_3 - n_3u_2) \\ &= \frac{m \cos \chi}{\sin \chi} \left\{ n_3u_2(u_1^2 + u_2^2 - 1) + n_2n_3(n_1u_1 + n_2u_2) \right. \\ &\quad \left. - u_2u_3(n_1u_1 + n_2u_2) + (1 - n_1^2 - n_2^2)n_2u_3 \right\} \\ &= \frac{m \cos \chi}{\sin \chi} \left\{ -n_3u_2u_3^2 - n_2n_3n_3u_3 + u_2u_3n_3u_3 + n_3^2n_2u_3 \right\} \\ &= 0. \end{aligned}$$

Similar calculations show that coefficients of  $\frac{1}{g_2}u_{1\xi_2}$ , and  $\frac{1}{g_i}u_{2\xi_i}$ ,  $\frac{1}{g_i}v_{1\xi_i}$  and  $\frac{1}{g_i}v_{2\xi_i}$ ,  $i = 1, 2$ , are all zero. The equation (5.7) becomes

$$(C.3) \quad n_{1t} = -\frac{v_3(n_3v_2 - n_2v_3)}{g_1u_3}m_{\xi_1} - \frac{v_3(n_2u_3 - n_3u_2)}{g_2v_3}m_{\xi_2}.$$

Using the expression (4.10) for the components of  $\mathbf{n}$ , we get (see the result (4.11)) the equation (5.9). Similarly the equation (5.10) can be derived.

#### APPENDIX D

We derive here the equation (6.8) from (6.7). If we differentiate the term in the bracket in (6.7) with respect to  $t$ , we shall have to evaluate  $(\sin \chi)_t$ . Since  $\sin^2 \chi = 1 - \cos^2 \chi$ , we get

$$(D.1) \quad \sin \chi (\sin \chi)_t = -\cos \chi (\cos \chi)_t = -\cos \chi \langle \mathbf{u}, \mathbf{v} \rangle_t.$$

Differentiation of  $\langle \mathbf{u}, \mathbf{v} \rangle$  contains term  $u_{3t}$  and  $v_{3t}$ . We use  $u_{3t} = -\frac{1}{u_3}(u_1u_{1t} + u_2u_{2t})$  and  $v_{3t} = -\frac{1}{v_3}(v_1v_{1t} + v_2v_{2t})$ . Then (D.1) gives

$$(D.2) \quad \sin \chi (\sin \chi)_t = -\cos \chi \left\{ \frac{1}{u_3}(u_3v_1 - u_1v_3)u_{1t} - \frac{1}{u_3}(u_2v_3 - u_3v_2)u_{2t} \right. \\ \left. - \frac{1}{v_3}(u_3v_1 - u_1v_3)v_{1t} + \frac{1}{v_3}(u_2v_3 - u_3v_2)v_{2t} \right\}.$$

Now we use (4.10) for eliminating  $u_3v_1 - u_1v_3$  and  $u_2v_3 - u_3v_2$  and get an expression for  $(\sin \chi)_t$

$$(D.3) \quad (\sin \chi)_t = \cos \chi \left( -\frac{n_2}{u_3}u_{1t} + \frac{n_1}{u_3}u_{2t} + \frac{n_2}{v_3}v_{1t} - \frac{n_1}{v_3}v_{2t} \right).$$

Differentiating (6.7) and using (D.3), we get the energy transport equation in the form (6.8).

## APPENDIX E. (NOTATIONS)

All variables are suitably non-dimensionalised.

$(\mathbf{x}, t) = (x_1, x_2, t)$	for 2-D KCL.
$(\mathbf{x}, t) = (x_1, x_2, x_3, t)$	for 3-D KCL.
$\Omega: \varphi(\mathbf{x}, t) = 0$	a surface in $(\mathbf{x}, t)$ space.
$\Omega_t: \varphi(\mathbf{x}, t) = 0, t = \text{constant},$	a moving surface in $\mathbf{x}$ -space at a fixed time $t$ .
$\Omega$	mean curvature of $\Omega_t$ .
$m$	normal velocity of $\Omega_t$ and is the metric associated with $t$ in ray coordinates.
$\mathbf{n}$	unit normal of $\Omega_t$ .
$\theta$	the angle $\mathbf{n}$ makes with $x$ -axis for 2-D KCL.
$\chi$	ray velocity = $m\mathbf{n}$ for isotropic evolution of $\Omega_t$ .
$(\xi, t)$	ray coordinates for 2-D KCL.
$g$	metric associated with $\xi$ .
$\mathbf{u}$	unit tangent vector to $\Omega_t$ for 2-D KCL.
$(\xi_1, \xi_2, t)$	ray coordinates for 3-D KCL.
$g_i$	metric associated with $\xi_i, i = 1, 2$ .
$\mathbf{u}, \mathbf{v}$	unit tangent vectors on $\Omega_t$ in direction of the coordinates $\xi_1$ and $\xi_2$ for 3-D KCL.
$\mathbf{L}$	$\nabla - \mathbf{n}\langle \mathbf{n}, \nabla \rangle$ .
$\chi$	$\cos \chi = \langle \mathbf{u}, \mathbf{v} \rangle, 0 < \chi < \pi$ .
$\mathcal{K}_t$	kink curve on $\Omega_t$ .
$(E_1, E_2)$	unit normal to the shock curve in $(\xi_1, \xi_2)$ -plane.
$K$	velocity of propagation of a shock curve in $(\xi_1, \xi_2)$ -plane.
$[f] := f_+ - f_-$	jump of a quantity across a shock curve in $(\xi_1, \xi_2)$ -plane.
$a_{ij}, b_{ij}^{(1)}, b_{ij}^{(2)}$	components of $7 \times 7$ matrices $B^{(1)}$ and $B^{(2)}$ in equation (6.12)
$w$	amplitude of a nonlinear wavefront in a polytropic gas (see relation (6.2))
$A$	ray tube area.
$\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}$	$n_1 \frac{\partial}{\partial x_3} - n_3 \frac{\partial}{\partial x_1}, n_3 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_3}$ .
$\mu_1, \mu_2 (= -\mu_1), \mu_3 = 0$	eigenvalues of the frozen equations (7.9)-(7.11).
$\nu_1, \nu_2 (= -\nu_1), \nu_3 = \dots = \nu_7 = 0$	eigenvalues of the frozen system (6.12) with $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')$ , where $\langle \mathbf{u}', \mathbf{v}' \rangle = 0$ .
$\lambda_1, \lambda_2 (= -\lambda_1), \lambda_3 = \dots = \lambda_7 = 0$	eigenvalues of the system (6.12).
$(\eta_1, \eta_2)$	orthogonal coordinates on $\Omega_t$ with unit tangent vectors $\mathbf{u}', \mathbf{v}'$ frozen at a point $P_0$ .
$(g'_1, g'_2)$	metrics associated with $\eta_1$ and $\eta_2$ frozen at a point $P_0$ .
$\gamma_1, \gamma_2, \delta_1, \delta_2, e_1, e_2, e'_1, e'_2$	coefficients occurring in section 8.3.

## ACKNOWLEDGEMENT

The authors sincerely thank Prof. Siddhartha Gadgil and Dr. Murali K. Vemuri for valuable discussions. The authors also express financial support for this work from CSIR (under the Scheme No. 21.(0641)05/EMR-II) and DST (under DST-DAAD joint research project No. INT/DAAD/P-141/2006). K. R. Arun would like to express his gratitude to the Council of Scientific & Industrial Research (CSIR) for supporting his research at the Indian Institute of Science under the grant-09/079(2084)/2006-EMR-1.

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