First Order Partial Differential Equations: a simple approach for beginners

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ABSTRACT: Usually a course on partial differential equations (PDEs) starts with the theory of first order PDEs, which turns out to be quite time consuming for a teacher and difficult for students due to dependence of the proofs on geometry of Monge curves and strips, and construction of an integral surface with their help. In this article, we present a simpler theory of first order PDEs using only the characteristic curves in the space of independent variables. In addition we discuss existence of special types of singularities along characteristic curves, a very important feature of first order PDEs.

1 Introduction and classification

1.1 Introduction

The classical theory of first order PDE started in about 1760 with Euler and D'Alembert and ended in about 1890 with the work of Lie. In the intervening period great mathematicians: Lagrange, Charpit, Monge, Pfaff, Cauchy, Jacobi and Hamilton made deep and important contributions to the subject and mechanics. Complete integral played a very important role in their work. Quote from Demidov " Lie developed the connection between 'groups of infinitely small transformations' and finite continuous groups of transformations in three theorems which make the foundation of the theory of Lie algebras. Lie discovered the connections while studying linear homogeneous PDEs of first order. Thus these equations came to the field on which the theory of Lie groups originally rooted itself." We realize that study of first order PDE lead to one of the most remarkable mathematical achievements of 19th century. Unfortunately, these mathematical theories are no longer treated as essential for study in a basic course in PDE [3].

Following presentation[1] of the theory of first order PDE as in Goursat (1917), Courant and Hilbert (1937, 1962), Sneddon (1957) and Garabedian (1964), the subject has been introduced to generations of students by defining an integral surface in the space of independent and dependent variables. This requires developing concepts of Monge curves[2] and Monge strips[2] leading to a system of ordinary differential equations, called Charpit [3] equations and a complicated geometrical proofs for existence and uniqueness of the solution of a Cauchy problem. We did follow this mathematically beautiful but not necessarily simple procedure in our book (Prasad and Ravindran (1985)) but now I feel that, in this approach, attention of students is unnecessarily diverted to geometrical concepts from main results of PDEs and at least three lecture hours are lost. Therefore, I have developed a theory of first order PDE using only the characteristic curves in the space of independent variables. In this article, we present this approach, which I have taught to a few groups of students.

I have omitted a special class of solutions known as complete integrals. Though complete integrals play important role in physics, the theory of complete integrals does not seem to be important in further development of the theory of first order PDEs and conservation laws. However, I have included here a discussion of *existence and propagation of singularities in the derivatives of the solution along characteristic curves.* These are important features of all hyperbolic PDEs [9, 10] but seldom discussed for the first order PDEs, which are simplest examples of hyperbolic equations. It is a bit difficult for undergraduate students to comprehend full implications of the usual definition of a generalised solution and hence, because of limited aim in this article, I have used a very simple function space to deal with discontinuities in the derivatives of the solution.

[1]Exception is Evans' book (1998) which is not a book for a first course but a comprehensive survey of modern techniques in theoretical study of PDE.

[2]Monge curves and Monge strips (in $(x_1, x_2, ..., x_m, u)$-space of independent variables $(x_1, x_2, ..., x_m)$ and dependent variable u) have been called characteristic curves and characteristic strips by all other authors but we reserve the word "characteristics" to be associated with the projections of Monge curves on the space of independent variables consistent with the use of this word for a higher order equation or a systems of equations.

[3]**Historical note:** In the method of characteristics of a first order PDE we use Charpit equations (1784) (see ([11]; for derivation see [10]). Unfortunately Charpit's name is not mentioned by Courant and Hilbert [1], and Garabedian [4]; and sadly even by Gaursat [5], who called these equations simply as *characteristic equations.* This may have occurred because Charpit died before he could follow up his manuscript sent to Paris Academy of Sciences. Later Lacroix published his results in 1814 (A.R. Forsyth, Treatise DE, 1885-1928) and finally Charpit's manuscript was found in the beginning of the 20th century. Charpit found these equation while trying to find complete integrals (see Demidov [2]). Those interested in teaching Charpit's method may consult M. Delgado, The Lagrange-Charpit Method, SIAM Review, **39**, 1997. **(My view: We need not give too much emphasis on Charpit's method in a course today.)**

1.2 Meaning of a first order PDE and its solution

In this article we shall consider u to be a real function of two *real* independent variables x and y. Let D be a domain in (x,y)-plane and u a *real* valued function defined on D:

$$u\colon D \to \mathbb{R},\ D \subset \mathbb{R}^2$$

Definition 1.1. A first order partial differential equation is a relation of the form

$$F(x, y, u, u_x, u_y) = 0, \tag{1.1}$$

where

$$F\colon D_3 \to \mathbb{R},\ D_3 \subset \mathbb{R}^5.$$

Note 1.2. D_3 is a domain[4] in $\mathbb{R}^5$ where the function F of five independent variables is defined.

Definition 1.3. *A classical (or genuine) solution* of the PDE (1.1) is a function $u\colon D \to \mathbb{R}, D \subset \mathbb{R}^2$ such that $u \in \mathcal{C}^1(D)$, $(x, y, u(x,y), u_x(x,y), u_y(x,y)) \in D_3$ when $(x,y) \in D$ and $F\left(x, y, u(x,y), u_x(x,y), u_y(x,y)\right) = 0$ for all $(x,y) \in D$.

Note 1.4. The problem of finding a solution u of (1.1) also involves finding conditions on F for the existence of the solution and finding the domain D where solution is defined.

Example 1.5. Since we are dealing only with real functions, the PDE

$$u_x^2 + u_y^2 + 1 = 0 \tag{1.2}$$

does not have any solution.

1.3 Classification of first order PDEs

When the function F in (1.1) is not linear in u, u_x and u_y, the equation (1.1) is called *nonlinear*. When F is linear in u_x, u_y, but not in u, the equation is of the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \tag{1.3}$$

where a and b depend also on u. This nonlinear equation is called a *quasilinear equation.* A first order *semilinear equation* is an equation of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u), \tag{1.4}$$

where nonlinearity in u appears only in the term on the right hand side. A *linear non-homogeneous* first order equation is of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y). \tag{1.5}$$

[4]In this article we denote by D a domain in $\mathbb{R}^2$ where a solution is defined, by D_1 a domain in $\mathbb{R}^2$ where the coefficients of a linear equation are defined and by D_2 is a domain in (x,y,u)-space i.e., $\mathbb{R}^3$.

1.4 Cauchy problem

A PDE in any area of application is always encountered with some auxiliary conditions. For a first order PDE this condition can be formulated in the form of a Cauchy problem, which we state in a simple language below.

Consider a curve γ in (x, y)-plane given by

$$\gamma\colon x = x_0(\eta),\ y = y_0(\eta),\ \eta \in I \subset \mathbb{R}, \tag{1.6}$$

where I is an interval of the real line. Assume that a function $u_0\colon I \to \mathbb{R}$ is given as $u_0(\eta)$. A *Cauchy problem* is to determine a solution $u(x, y)$ of (1.1) in a neighbourhood of γ such that it takes the prescribed value $u_0(\eta)$ on γ, i.e,

$$u(x_0(\eta), y_0(\eta)) = u_0(\eta),\ \forall\ \eta \in I. \tag{1.7}$$

2 Linear and semilinear Equations

2.1 Preliminaries through an example

Let us start with the simplest PDE, namely the transport equation in two independent variables. Consider the PDE

$$u_y + cu_x = 0,\ c = \text{real constant}. \tag{2.1}$$

Introduce a variable $\eta = x - cy$. For a fixed η, $x - cy = \eta$ is a straight line with slope $\frac{1}{c}$ in (x, y) plane. Along this straight line

$$x(y) = cy + \eta \tag{2.2}$$

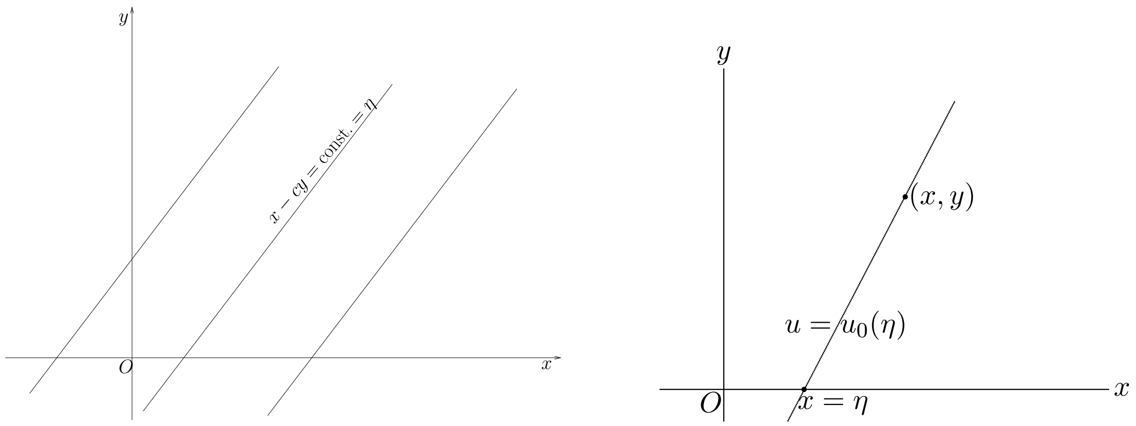
the derivative of a solution $u(x, y) = u(x(y), y)$ is

$$\begin{aligned}\frac{d}{dy}u(x(y), y) &= u_x\frac{dx}{dy} + u_y \\ &= cu_x + u_y \\ &= 0.\end{aligned}$$

Thus, the solution u is constant along curves $x - cy = \eta$, as seen in Figure 1.

Remark 2.1. The curves $x - cy = \text{constant}$ are known as *characteristic curves* (see the definition (2.3) later) of the PDE (2.1).

The characteristic curves carry constant values of the solution u.



(a) (b)

Figure 1: Characteristic curves of $u_y + cu_x = 0$. (a) - Characteristics form a one parameter family of straight lines $x - cy =$ constant. (b) - u is constant along a characteristic curve.

Consider an initial value problem, a Cauchy problem, in which

$$u(x, 0) = u_0(x). \tag{2.3}$$

To find solution at (x, y), draw the characteristic through (x, y) and let it meet the x-axis at $x = \eta$. Then, u is constant on $x - cy = \eta$, i.e,

$$\begin{aligned} u(x, y) &= u(\eta, 0) \\ &= u_0(\eta). \end{aligned}$$

Hence, as $\eta = x - cy$,

$$u(x, y) = u_0(x - cy). \tag{2.4}$$

Remark 2.2. Instead of prescribing the value of u on the x-axis, we can prescribe it on any curve in the (x, y)-plane, say on the curve $\gamma : x = x_0(y),\ x_0 \in \mathcal{C}^1(I)$, as shown in the Figure 2(a). Equation of this curve can be parametrically represented as $\gamma : x = x_0(\eta), y = \eta$. In order that the above method of construction of a solution (with arbitrary data on γ) works, we cannot choose the datum curve γ arbitrarily. A necessary condition is that it should not coincide with a characteristic line $x - cy =$ constant (see the situation discussed at the end of this example) or it should not be a curve having a characteristic line as a tangent at any point of it[5].

If the characteristic through (x, y) meets $\gamma : x = x_0(\eta), y = \eta$ at $(x_0(\eta), \eta)$ as shown in Figure 2(a), then

$$x - cy = x_0(\eta) - c\eta. \tag{2.5}$$

[5]When γ is tangent to a characteristic at a point P, then the Cauchy data on γ on one side of P interferes with that on the other side of P, see problem 1 in section 2.3 Problem set.

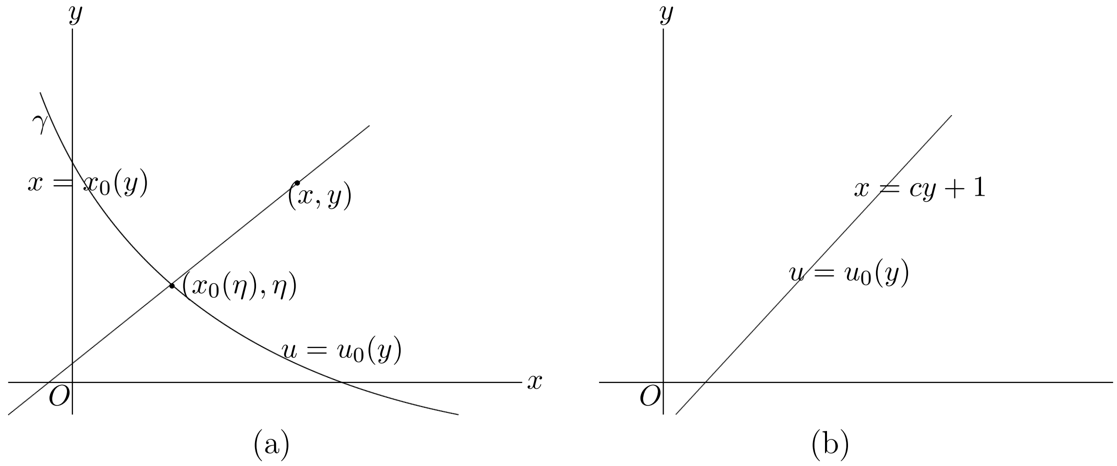


Figure 2: (a) - The datum curve $x = x_0(y)$, i.e. $x = x_0(\eta), y = y_0(\eta) = \eta$, is nowhere tangential to a characteristic curve. (b) - The datum curve is a characteristic curve $x = cy + 1$.

Suppose γ is nowhere tangential to a characteristic curve, then $x_0'(\eta) \neq c$ and, using inverse function theorem, we can solve this equation for η locally in a neighbourhood of *each point* of γ in the form

$$\eta = g(x - cy), \quad g \in \mathcal{C}^1(I_1), \quad I_1 \subset I. \tag{2.6}$$

Hence, the solution of this noncharacteristic Cauchy problem in a domain containing the curve γ is

$$\begin{aligned} u(x, y) &= u_0(\eta) \\ &= u_0\left(g(x - cy)\right). \end{aligned} \tag{2.7}$$

In Figure 2(b) let the datum curve be a characteristic curve $x = cy + 1$. The data $u_0(y)$ prescribed on this line must be a constant, say $u_0(y) = a$. Now we can verify that the solution is given by

$$u(x, y) = a + (x - cy - 1)h(x - cy), \tag{2.8}$$

where $h(\eta)$ is an *arbitrary* $\mathcal{C}^1$ function of just one argument. This verifies a general property that the solution of a characteristic Cauchy problem, when it exists, it is *not unique.*

2.2 Theory of linear and semilinear equations

Linear and semilinear equations can be treated together. We take

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u). \tag{2.9}$$

where a, b, c are $\mathcal{C}^1$ functions of their arguments. The operator $a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ on the left hand side of this equation represents differentiation in a direction (a, b)

at the point (x, y) in (x, y)-plane. Let us consider a curve, whose tangent at each point (x, y) has the direction (a, b) . Coordinates $(x(\sigma), y(\sigma))$ of a point on this curve satisfy

$$\frac{dx}{d\sigma} = a(x, y), \ \frac{dy}{d\sigma} = b(x, y) \tag{2.10}$$

or

$$\frac{dy}{dx} = \frac{a(x, y)}{b(x, y)}. \tag{2.11}$$

General solution of (2.10) or (2.11) contains an arbitrary constant and gives a one parameter family of curves in (x, y)-plane. Along a particular curve of this family, variation of a solution of (2.9) is given by

$$\begin{aligned} \frac{du}{d\sigma} &= \frac{dx}{d\sigma}u_x + \frac{dy}{d\sigma}u_y \\ &= au_x + bu_y. \end{aligned}$$

Using (2.9) we get

$$\frac{du}{d\sigma} = c(x, y, u). \tag{2.12}$$

Definition 2.3. A curve given by (2.10) or (2.11) is called a *characteristic curve* of the PDE (2.9). The equation (2.12) is called a *compatibility condition*.

Remark 2.4. The characteristic curves of linear or semilinear equation(2.9) form a one parameter family of curves. Every solution of the PDE (2.9) satisfies the compatibility condition (2.12) along any of these characteristic curves.

2.2.1 Solution of a Cauchy problem

The Cauchy problem has been stated at the end of section 1. We present here an algorithm[6] to solve it for the equation (2.9). We assume that γ cuts each one of the characteristic curves transversally. We first solve an initial value problem for ordinary differential equations (ODEs)

$$\frac{dx}{d\sigma} = a(x, y), \tag{2.13}$$

$$\frac{dy}{d\sigma} = b(x, y), \tag{2.14}$$

$$\frac{du}{d\sigma} = c(x, y, u) \tag{2.15}$$

[6]The proof of the theorem (3.4) shows that this algorithm indeed works. Existence and uniqueness of a solution of the Cauchy problem would require conditions on the functions a, b and c appearing in the equation (2.9) and x_0, y_0 and u_0 appearing in the Cauchy data (1.6). The theorem (3.4) gives sufficient conditions for the existence and uniqueness of the solution for a more general equation, namely quasilinear equation.

with the initial data

$$x = x_0(\eta), \ y = y_0(\eta), \ u = u_0(\eta) \text{ at } \sigma = 0. \tag{2.16}$$

Solving (2.13) and (2.14) with initial data $x = x_0(\eta), y = y_0(\eta)$ at $\sigma = 0$ we get

$$x = X(\sigma, \eta), \ y = Y(\sigma, \eta). \tag{2.17}$$

Then solving (2.15) with $u = u_0(\eta)$ at $\sigma = 0$ we get

$$u = U(\sigma, \eta). \tag{2.18}$$

Solving for σ and η from (2.17) as functions of x and y and substituting in (2.18) we get the solution

$$u(x, y) = U(\sigma(x, y), \eta(x, y)) \tag{2.19}$$

of the PDE in a neighbourhood of γ.

Remark 2.5. The system of ODEs (2.13) and (2.14) is nonlinear. Hence we expect only a local existence of characteristic curves. The ODE (2.15) is also nonlinear. Hence, even if we get global solution of (2.13) and (2.14) and the solution of the initial value problem for (2.15) exists, we expect to get solution of (2.15) along the characteristic curve only locally i.e, in a neighbourhood of $\sigma = 0$, which corresponds to a point on γ.

Example 2.6. Consider the equation

$$yu_x - xu_y = 0. \tag{2.20}$$

The characteristic equations are

$$\begin{aligned} \frac{dx}{y} &= \frac{dy}{-x} \\ \implies xdx + ydy &= 0 \end{aligned} \tag{2.21}$$

which gives characteristic curves

$$x^2 + y^2 = \text{constant} = a^2, \ \text{say}. \tag{2.22}$$

These are circles with centre at the origin and radius a.

Let us consider a Cauchy problem for (2.20) in which Cauchy data is prescribed on the x-axis as

$$u(x, 0) = u_0(x). \tag{2.23}$$

The compatibility condition on the characteristic curves is

$$\frac{du}{dx} = 0 \tag{2.24}$$

which means u is constant on the circles (2.22). This implies that for the existence of the solution, u_0 must satisfy a condition $u_0(x) = u_0(-x)$, i.e, the initial data must be an even function. Let us choose

$$u_0(x) = x^2. \tag{2.25}$$

Now the solution exists and, since all solutions u are constant on the circles (2.22), it is uniquely determined as

$$u(x, y) = x^2 + y^2. \tag{2.26}$$

Example 2.7. It is interesting to take another problem which has explicit solution. Consider the Cauchy problem (to be defined later):

$$u_x = c_1(x, y)u + c_2(x, y), \;\; u(0, y) = u_0(y), \tag{2.27}$$

where $u_0(y)$, $c_1(x, y)$ and $c_2(x, y)$ are known continuous functions. It is easy to derive (as if it is a problem in ODE) the unique solution of this problem:

$$u(x, y) = u_0(y)e^{\int_0^x c_1(\sigma,y)d\sigma} + e^{\int_0^x c_1(\sigma,y)d\sigma} \int_0^x c_2(\zeta, y)e^{\int_0^\zeta c_1(\sigma,y)d\sigma} d\zeta \tag{2.28}$$

We again ask a question. Is so easy to solve a Cauchy problem of a first order PDE? The answer is yes. Every non-characteristic Cauchy problem for a linear non-homogeneous equation can be reduced to the Cauchy problem (2.7).

To prove the above assertion, consider a non-characteristic Cauchy problem for the nonhomogeneous linear first order PDE (1.5) with Cauchy data data prescribed on a smooth curve written in the form

$$\gamma : \quad \psi(x, y) = 0, \tag{2.29}$$

Let the one parameter family of characteristic curves, obtained by solving the equations (2.10), be represented by

$$\varphi(x, y) = constant \tag{2.30}$$

then φ satisfies

$$(a\varphi_x + b\varphi_y) = 0. \tag{2.31}$$

For a non-characteristic cuve, the functions $\varphi(x, y)$ and $\psi(x, y)$ are independent and we choose these functions as coordinates in place of x and y. Using (2.31), we find that the transformed equation for u becomes

$$(a\psi_x + b\psi_y)u_\psi = c_1(x, y)u + c_2(x, y). \tag{2.32}$$

In terms of functions $U(\varphi, \psi) \doteq u(x, y), C_1(\varphi, \psi) \doteq c_1(x, y)$ and $C_2(\varphi, \psi) \doteq c_2(x, y)$, we get a new Cauchy problem

$$U_\psi(\varphi, \psi) = C_1(\varphi, \psi)U + C_2(\varphi, \psi), \quad U(\varphi, 0) = U_0(\varphi). \tag{2.33}$$

We have shown in example 7 that this problem can be solved uniquely. The solution for $U(\varphi, \psi)$ will only be local, because the the characteristic equations (2.10) are nonlinear and also the coordinates φ and ψ can be introduced only locally.

2.3 Problem set

1. Show that a characteristic of the equation $u_x - u_y = 0$ touches the branch of the hyperbola $xy = 1$ in the first quadrant of the (x, y)-plane at the point $P(1, 1)$. Verify that P divides the hyperbola into two portions such that the Cauchy data prescribed on one portion determines the value u on the other portion.

2. Find the characteristics of the equations

   (i) $2xyu_x - (x^2 + y^2)u_y = 0$,

   (ii) $(x^2 - y^2 + 1)u_x + 2xyu_y = 0$.

3. Show that if u is prescribed on the interval $0 \leq y \leq 1$ of the y-axis, the solution of 2(ii) is completely determined in the upper half of the (x, y)-plane.

4. Find the solutions of the following Cauchy problems and the domains in which they are determined in the (x, y)-plane:

   (i) $yu_x + xu_y = 2u$ with $u(x, 0) = f(x)$ for $x > 0$,

   (ii) $yu_x + xu_y = 2u$ with $u(0, y) = g(y)$ for $y > 0$,

   (iii) $u_x + u_y = u^2$ with $u(x, 0) = 1$ for $-\infty < x < \infty$.

3 Quasilinear equations

3.1 Derivation of characteristic equations and compatibility condition

It is simple to discuss the theory of a quasilinear equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{3.1}$$

using the theory of linear equations. We assume that a, b, c are $\mathcal{C}^1(D_2)$.

Consider a known solution $u(x, y)$ of (3.1). Let us substitute the function $u(x, y)$ for u in the coefficients a,b and c, then the coefficients $a(x, y, u(x, y)), b(x, y, u(x, y))$ and $c(x, y, u(x, y))$ become known functions of x and y, and u satisfies a relation in which a and b are functions of x and y only. Following the steps of the last section, we find that the following ODEs are satisfied simultaneously

$$\begin{aligned} \frac{dx}{d\sigma} &= a(x, y, u(x, y)), \\ \frac{dy}{d\sigma} &= b(x, y, u(x, y)), \\ \frac{du}{d\sigma} &= c(x, y, u(x, y)). \end{aligned} \tag{3.2}$$

Now, this is true for every solution $u(x,y)$. Hence, it follows that for an arbitrary solution of (3.1), there are characteristic curves in the (x,y)-plane given by

$$\frac{dx}{d\sigma} = a(x,y,u), \ \frac{dy}{d\sigma} = b(x,y,u) \tag{3.3}$$

and along each of these characteristic curves the compatibility condition

$$\frac{du}{d\sigma} = c(x,y,u) \tag{3.4}$$

must be satisfied.

Remark 3.1. In the case of linear and semilinear equations, the characteristic equations are independent of u. Hence, the characteristic curves are completely determined by (2.13) and (2.14) without any reference to the solution u. There is only one characteristic through a point (x_0, y_0), provided $a, b \in \mathcal{C}^1$ in a neighbourhood of (x_0, y_0) and $a^2(x_0,y_0)+b^2(x_0,y_0) \neq 0$. *Thus the characteristic curves of a semilinear equation form a one parameter family of curves arising out of one constant of integration of the ODE (2.11).* On the contrary, the characteristic equations and the compatibility conditions of a quasilinear equation form a coupled system of three equations. Hence, in this case, through any point (x_0, y_0) a distinct characteristic curve is determined for a given solution. Through the same point (x_0, y_0), there are infinity of tangent directions $dy/dx = b(x,y,u)/a(x,y,u)$ of characteristic curves depending on the value u_0 of u there. *Thus, the characteristic curves of a quasilinear equation form a two parameter family of curves arising out of two constants of integration of a pair of ODEs* $\frac{dy}{dx} = b(x,y,u)/a(x,y,u)$, $\frac{du}{dx} = b(x,y,u/a(x,y,u)$.

3.2 Solution of a Cauchy problem

As an algorithm to solve a Cauchy problem with data given in (1.6)-(1.7), we first solve the system of three equations (3.3)-(3.4) simultaneously with initial data

$$x = x_0(\eta), \ y = y_0(\eta), \ u = u_0(\eta) \text{ at } \sigma = 0. \tag{3.5}$$

This gives the solution in the form (2.17)-(2.18). As before, we try to solve σ and η as functions of x and y from (2.17) and substitute them in (2.18) to get the solution (2.19) of the Cauchy problem. We explain the method with the help of a simple example.

Example 3.2. Consider the equation

$$uu_x + u_y = 0 \tag{3.6}$$

with the Cauchy data

$$u(x,0) = x, \ 0 \leq x \leq 1$$

which can be put in the form

$$x = \eta, \ y = 0, \ u = \eta, \ 0 \le \eta \le 1. \tag{3.7}$$

Solving the characteristic equations and the compatibility condition

$$\frac{dx}{d\sigma} = u, \ \frac{dy}{d\sigma} = 1, \ \frac{du}{d\sigma} = 0$$

with initial data (3.7) at $\sigma = 0$ we get

$$x = \eta(\sigma + 1), \ y = \sigma, \ u = \eta. \tag{3.8}$$

The characteristic curve passing through a point $x = \eta$ on the x-axis is a straight line $x = \eta(y + 1)$. These characteristics for all admissible but fixed values of η, i.e, $0 \le \eta \le 1$ pass through the same point $(0, -1)$ and cover the wedged shaped portion D of the (x, y)-plane bounded by two extreme characteristics $x = 0$ and $x = y + 1$. Note that $u = \eta$ in (3.8) shows that u is constant on each of these characteristics, being equal to the abscissa of a point where a characteristic intersects the x-axis. The solution is determined in the wedged shaped region D as shown in the Figure 3. Eliminating σ and η from (3.8), we get the solution of the Cauchy problem as

$$u = \frac{x}{y + 1}. \tag{3.9}$$

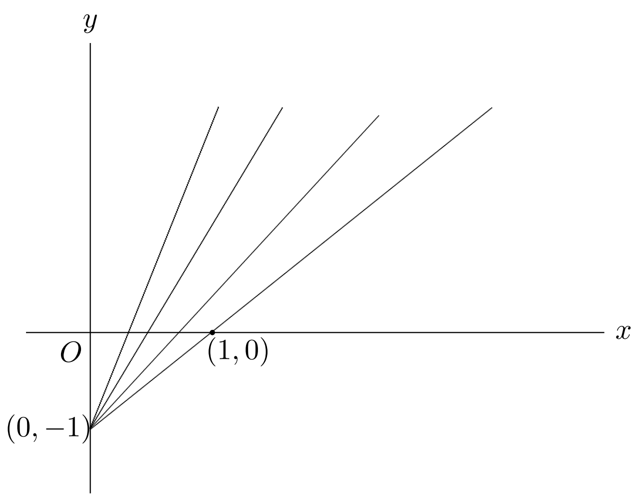


Figure 3: The solution is determined in the wedged shaped region D of the (x, y)-plane.

Remark 3.3. We note three very important aspects of the quasilinear equations from this example.

(i) The domain D in the (x, y)-plane in which the solution is determined depends on the data prescribed in the Cauchy problem. Had we prescribed a constant value of $u(x, 0)$, say $u(x, 0) = \frac{1}{2}$ for $0 \le x \le 1$, the characteristics would have been a family of parallel straight lines $y - 2x = -2\eta$ and the domain D would have been the infinite strip bounded by the extreme characteristics $y - 2x = 0$ and $y - 2x = -2$ as shown in the Figure 4.

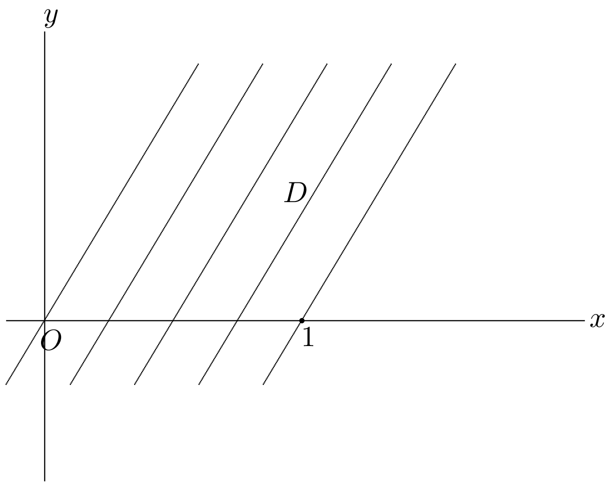


Figure 4: The domain D when the Cauchy data is $u(x, 0) = \frac{1}{2}$ for $0 \le x \le 1$.

(ii) Even though the coefficients in the equation (3.6) and Cauchy data (3.7) are regular, the solution develops a singularity at the point $(0, -1)$. Geometrically, this is evident from the fact that the characteristics which carry different value of u all intersect at $(0, -1)$. Analytically, this is clear from the explicit form of the solution (3.9). The appearance of a singularity in the solution of a Cauchy problem even for a smooth Cauchy data is a property associated with nonlinear differential equations.

(iii) This example also highlights the fact that the solution of quasilinear equation is only local i.e., we expect the solution to be valid only in a neighbourhood of γ. In the case of the solution (3.9), the neighbourhood is quite large, it is the wedged shaped region D shown in the Figure 3.

We now prove a theorem showing local existance and uniqueness of a solution of a noncharacteristic Cauchy problem for the equation (3.1).

Theorem 3.4. *Consider a Cauchy problem for the PDE (3.1) with Cauchy data* $u_0(\eta)$ *prescribed on a curve* γ *given by (1.6), where* I *is an open interval, say for* $0 < \eta < 1$. *Let*
(i) $x_0(\eta), y_0(\eta), u_0(\eta) \in C^1(I)$,
(ii) $a(x, y, u), b(x, y, u)$ *and* $c(x, y, u)$ *be* C^1 *functions on a domain*[7] D_2 *of* $\mathbb{R}^3$,

[7]for an explanation of the symbol D_2, see the footnote number 3.

(iii) the curve Γ: $x = x_0(\eta), y = y_0(\eta), u = u_0(\eta)$ for $\eta \in (0,1)$ belongs to D_2 and
(iv) the transversality condition

$$\frac{dy_0(\eta)}{d\eta} a(x_0(\eta), y_0(\eta), u_0(\eta)) - \frac{dx_0(\eta)}{d\eta} b(x_0(\eta), y_0(\eta), u_0(\eta)) \neq 0 \tag{3.10}$$

is satisfied on the curve γ.

Then there exists a unique solution $u(x, y)$ of the Cauchy problem in a neighourhood D of γ.

Remark 3.5. The condition (3.10) implies that

$$\left(\frac{dx_0(\eta)}{d\eta}\right)^2 + \left(\frac{dy_0(\eta)}{d\eta}\right)^2 \neq 0$$

so that γ is free from a singularity. It also implies that γ is not a characteristic curve.

Proof. Since a, b, c have continuous partial derivatives with respect to x, y, u, the system of ODEs (3.3) and (3.4) have unique continuously differentiable solution with respect to σ [7] of the form

$$x = x(\sigma, \eta),\ y = y(\sigma, \eta),\ u = u(\sigma, \eta) \tag{3.11}$$

satisfying the initial condition

$$x(0, \eta) = x_0(\eta),\ y(0, \eta) = y_0(\eta),\ u(0, \eta) = u_0(\eta) \text{ for an } \eta \in I. \tag{3.12}$$

Consider now a point P_0 on γ corresponding to $\eta = \eta_0$. As $x_0(\eta), y_0(\eta), u_0(\eta)$ are continuously differentiable, the functions in the solution (3.11) are continuously differentiable functions of two independent variables σ and η [7] in a domain of (σ, η)-plane containing $(0, \eta_0)$. In view of our assumption (3.10), the Jacobian

$$\frac{\partial(x, y)}{\partial(\sigma, \eta)} \equiv \begin{vmatrix} x_\sigma & x_\eta \\ y_\sigma & y_\eta \end{vmatrix} = a y_\eta - b x_\eta \tag{3.13}$$

does not vanish at $\sigma = 0$ for $0 < \eta < 1$ and in particular at the point $(0, \eta_0)$ in (σ, η)-plane.

Using the inverse function theorem we can uniquely solve for σ and η from the first two relations in (3.11) in terms of x and y

$$\sigma = \sigma(x, y),\ \eta = \eta(x, y) \tag{3.14}$$

in a neighbourhood D_{P_0} of P_0 in (x, y)-plane. Moreover, the functions σ and η in (3.14) are $\mathcal{C}^1$ in D_{P_0}. Substitute (3.14) in the third relation in (3.11) to get a function $u(x, y)$ of x and y in D_{P_0}:

$$u(x, y) = u(\sigma(x, y), \eta(x, y)). \tag{3.15}$$

Further, this function is $\mathcal{C}^1$ on a domain D_{P_0} in (x, y)-plane. Note that at any point of the intersection of datum curve γ and the domain D_{P_0}

$$\begin{aligned} u(x_0(\eta), y_0(\eta)) &= u(\sigma(x_0, y_0), \eta(x_0, y_0)) \\ &= u(0, \eta) \\ &= u_0(\eta) \end{aligned}$$

which shows that the initial condition (3.10) is satisfied. Moreover,

$$\begin{aligned} au_x + bu_y &= a\,(u_\sigma \sigma_x + u_\eta \eta_x) + b\,(u_\sigma \sigma_y + u_\eta \eta_y) \\ &= u_\sigma(a\sigma_x + b\sigma_y) + u_\eta(a\eta_x + b\eta_y) \\ &= u_\sigma(\sigma_x x_\sigma + \sigma_y y_\sigma) + u_\eta(\eta_x x_\sigma + \eta_y y_\sigma), \text{ using (3.3)}, \\ &= u_\sigma \sigma_\sigma + u_\eta \eta_\sigma \\ &= u_\sigma \\ &= c. \end{aligned} \tag{3.16}$$

Hence, u satisfies the PDE (3.1) in D_{P_0}. Let $D = \{\bigcup_{P_0 \in \gamma} D_{P_0}\}$. We define u in D with the help of values of u in D_{P_0}. Now, we have solved the Cauchy problem in a neighbourhood D of γ.

For uniqueness of the solution, we note that each step of the construction of the function $u(x, y)$ in (3.13), namely solution of the ODEs (3.2), inversion of the functions leading to $\sigma(x, y)$ and $\eta(x, y)$ and substitution in $u(\sigma, \eta)$ all lead to unique results. Hence, the procedure leading to the construction of the solution (3.15) gives a unique solution of the Cauchy problem in the neighbourhood D_{P_0} of the point $(x_0(\eta_0), y_0(\eta_0))$ on γ and hence in the domain D containing γ. Is there any other method which may lead to another solution? Suppose there is another solution u_1. Then u_1 will also satisfy the same ODEs (3.2) with same initial data (3.12) and hence will coincide with u in each step of the proof of the theorem. Hence $u_1 = u$ in D. □

At this stage we need to comment on the possible cases of **non-exitance and non-uniqueness of the solution of a Cauchy problem**. We start with an observation that for a quasilinear equation whether a datum curve is a characteristic or is tangential to a characteristic depends on the Cauchy data $u_0(\eta)$. Now we first note that if the datum curve γ is tangential to a characteristic curve at some point, the right hand side of (3.11) vanishes at that point and the proof of existence of the solution breaks down in the neigbourhood of that point. The second case is when the datum curve γ coincides with a characteristic curve. In this case the data $u_0(\eta)$ can not be arbitrarily prescribed on γ and in order that the solution exists it must satisfy the compatibility condition (3.5). We shall show in Theorem 3.9, that if the compatibility condition is satisfied, there exists infinity of solutions.

3.3 Some more results for quasilinear equations

Remark 3.6. We mention below some important results on first order PDEs, refer to [10] more details.

1. A *first integral* of the characteristic equations (3.3) and the compatibility condition in (3.4) can be written in the form

$$\varphi(x, y, u) = c, \tag{3.17}$$

   where c is an arbitrary constant. If we can solve (3.17) for u to get $u = u(x, y, c)$, we get a *one parameter family of solutions* of the PDE (3.1).

2. Let $\varphi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ be two first integrals of the ODEs (3.3) and (3.4). Then *a solution* of (3.1) can be obtained by solving u from a relation

$$h(\varphi(x, y, u), \psi(x, y, u)) = 0 \tag{3.18}$$

   where h is an arbitrary $\mathcal{C}^1$ function of two arguments.

3. For *any given* function h, a solution $u(x, y)$ of (3.18) satisfies the PDE (3.1). In this sense (3.18), with arbitrary h, is called a *general solution* of (3.1). Given a Cauchy problem, it is easy to find out the function h such that (3.18) would give u which will be the solution of the Cauchy problem.

Now we prove an important theorem.

Theorem 3.7. *If $\phi(x, y)$ and $\psi(x, y)$ are two solutions of a quasilinear equation (3.1) in a domain D and they have a common value u_0 at a point $(x_0, y_0) \in D$, then characteristics of both solutions through (x_0, y_0) coincide in D and both solutions have same values on this common characteristic.*

Proof. Since the coefficients a, b, c are C^1 functions, there exists a unique solution

$$x = x(x_0, y_0, \sigma), \; y = y(x_0, y_0, \sigma), \; u = u(x_0, y_0, \sigma) \tag{3.19}$$

of the ODEs (3.3) and (3.4) with initial data $(x, y, u) = (x_0, y_0, u_0)$ at $\sigma = 0$.

The first two equations in (3.19) give the common characteristic through the point (x_0, y_0) for the two solutions ϕ and ψ and the third equation in it gives the common value of the solution on this characteristic.

This completes the proof of the theorem. □

Example 3.8. For the equation (3.5), the result stated in the theorem can be verified for the solutions $u = \frac{x}{y+1}$ (see (3.9)) and $u = \frac{1}{2}$ (see Remark 3.3.(i)) along the common characteristic $y = 2x - 1$. This also clarifies the non-uniqueness of the solution of a characteristic Cauchy problem, which we state in the form of a theorem.

Theorem 3.9. *A characteristic Cauchy problem, when the solution exist, has infinity of solutions.*

Proof. Assume that the compatibility condition (3.4) is satisfied on datum curve γ, which is a characteristic curve.

Choose a point P on γ. We take another curve γ_1 through the point P and we prescribe a smooth Cauchy data u_{10} on in such a way that it is not a characteristic curve and $u_{10}(P) = u_0(P)$. The solution u_1 of (3.1) with this Cauchy data on γ_1 exists and the original curve γ is a characteristic curve of the solution u_1. We can choose γ_1 in an infinity of ways and set up a characteristic Cauchy problem with it. Thus we get infinity of solutions with γ having data u_0 on it as a characteristic curve for each one of these solutions. □

3.4 Problem set

1. Show that all the characteristic curves of the partial differential equation

$$(2x + u)u_x + (2y + u)u_y = u$$

   through the point $(1, 1)$ are given by the same straight line $x - y = 0$.

2. Discuss the solution of the differential equation

$$uu_x + u_y = 0, \ \ y > 0, \ \ -\infty < x < \infty$$

   with Cauchy data

$$u(x, 0) = \begin{cases} \alpha^2 - x^2 & \text{for } \ |x| \leq \alpha, \\ 0 & \text{for } \ |x| > \alpha. \end{cases}$$

4 First order nonlinear equations in two independent variables

4.1 Derivation of Charpit's equations

The most general form of first order equation is (1.1), i.e.,

$$F(x, y, u, p, q) = 0; \ \ p = u_x, q = u_y, \tag{4.1}$$

where we assume that $F \in \mathcal{C}^2(D_3)$ with D_3 as a domain in $\mathbb{R}^5$.

With a general expression F in the PDE (4.1), there is no indication that we can find a transport equation for a solution u along a curve in (x, y)- plane. Therefore, it is really remarkable and interesting ([5], sections 83 and 85) that the discovery of such a curve was made by Lagrange and Charpit in an attempt to find a complete integral and by Cauchy in a geometric formulation with the help of Monge Cone [1]. Though this has a very rich history, it is quite time consuming for teaching in a class. We proceed differently by taking a known $\mathcal{C}^2$ solution $u(x, y)$ and differentiate (4.1) with respect to x. This gives

$$F_x + u_x F_u + p_x F_p + q_x F_q = 0.$$

Using $q_x = u_{yx} = u_{xy} = p_y$ and rearranging the terms we get

$$F_p p_x + F_q p_y = -F_x - pF_u \tag{4.2}$$

in which the quantities F_p, F_q are now known functions of x and y. This is a beautiful result, the first x-derivative of u, namely p is differentiated in the direction (F_p, F_q) in (x, y)-plane. Thus, for the known solution $u(x, y)$, consider a one parameter family of curves in (x, y)-plane given by

$$\frac{dx}{d\sigma} = F_p, \ \frac{dy}{d\sigma} = F_q. \tag{4.3}$$

Along these curves, (4.2) gives

$$\frac{dp}{d\sigma} = -F_x - pF_u. \tag{4.4}$$

Similarly, differentiating (4.1) with respect to y, we find that along the same curves given by (4.3), we have

$$\frac{dq}{d\sigma} = -F_y - qF_u. \tag{4.5}$$

The rate of change of u along these curves is

$$\begin{aligned} \frac{du}{d\sigma} &= u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma} \\ &= pF_p + qF_q. \end{aligned} \tag{4.6}$$

Note that (4.3)-(4.6) form a complete system of five ODEs for five quantities x, y, u, p and q irrespective of the solution $u(x, y)$ we take for their derivation. Thus, we find a beautiful set of equations, called *Charpit's equations* consisting of two *characteristic equations* (4.3) and three compatibility conditions (4.4)-(4.6). Given a set of values (u_0, p_0, q_0) at any point (x_0, y_0), so that $(x_0, y_0, u_0, p_0, q_0) \in D_3$, we can find a solution of Charpit's equations. Since the system is autonomous, the set of solutions of the Charpit's equations form a four parameter family of curves in (x, y)-plane.

Remark 4.1. Every solution $(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma))$ of the Charpit's equations satisfies the *strip condition*

$$\frac{du}{d\sigma} = p(\sigma)\frac{dx}{d\sigma} + q(\sigma)\frac{dy}{d\sigma} \tag{4.7}$$

on the curve $(x(\sigma), y(\sigma), u(\sigma))$ in (x, y, u)-space. For a geometrical interpretation of the strip condition, see [1, 10].

There is an interesting result, which says that *the function F is constant on any integral curve of Charpit's equations in D_3*. The proof is very simple

$$\frac{dF}{d\sigma} = \frac{dx}{d\sigma}F_x + \frac{dy}{d\sigma}F_y + \frac{du}{d\sigma}F_u + \frac{dp}{d\sigma}F_p + \frac{dq}{d\sigma}F_q$$

which vanishes when we use the Charpit's equations. This means that though not every solution of the Charpit's equations satisfies $F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) = 0$, if we choose u_0, p_0 and q_0 at (x_0, y_0) such that $F(x_0, y_0, u_0, p_0, q_0) = 0$, then $F = 0$ for all values of σ.

Definition 4.2. A set of five ordered functions $(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma))$ satisfying the Charpit equations (4.3) - (4.6) and $F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) = 0$ is called a *Monge strip* of the PDE (4.1).

The condition $F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) = 0$ imposes a relation between the four parameters of the set of all solutions of the Charpit's equations. Therefore the of Monge strips form a three parameter family of strips in (x, y, u)-space.

Definition 4.3. Given a Monge strip $(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma))$, the base curve in (x, y)-plane given by $(x(\sigma), y(\sigma))$ is called a *characteristic curve* of the PDE (4.1).

Remark 4.4. Characteristic curves of a linear first order PDE form a one parameter family of curves in (x, y)-plane. Those for a quasilinear equation form a two parameter family of curves. Finally those of a nonlinear equation (4.1) form a three parameter family of curves.

4.2 Solution of a Cauchy problem

We shall first state an algorithm to solve a Cauchy problem. Next we shall state a theorem which guarantees that this algorithm indeed gives a unique solution of the Cauchy problem.

For the nonlinear PDE (4.1), we need to solve the Charpit's equations (4.3)-(4.6) and for this we need initial values $p_0(\eta)$ and $q_0(\eta)$ on γ in addition to the values $x_0(\eta), y_0(\eta)$ and $u_0(\eta)$ given in the Cauchy data (1.6)-(1.7). Firstly we note that these initial values must satisfy the PDE, i.e,

$$F(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)) = 0. \tag{4.8}$$

Further, differentiating (1.7) with respect to η we get one more relation

$$p_0(\eta)x_0'(\eta) + q_0(\eta)y_0'(\eta) = u_0'(\eta). \tag{4.9}$$

We take two functions $p_0(\eta)$ and $q_0(\eta)$ satisfying (4.8)-(4.9) and complete the initial data for (4.3)-(4.6) at $\sigma = 0$ as

$$\begin{gathered} x(0,\eta) = x_0(\eta),\ y(0,\eta) = y_0(\eta),\ u(0,\eta) = u_0(\eta) \\ p(0,\eta) = p_0(\eta),\ q(0,\eta) = q_0(\eta). \end{gathered} \tag{4.10}$$

Now we solve the Charpit's equations (4.3)-(4.6) with initial data (4.10) and obtain

$$x = X(\sigma,\eta),\ y = Y(\sigma,\eta),\ u = U(\sigma,\eta),\ p = P(\sigma,\eta),\ q = Q(\sigma,\eta). \tag{4.11}$$

From the first two relations in (4.11), we solve σ and η as functions of x and y and substitute in the third relation to get the solution $u(x, y)$ of the Cauchy problem.

The following theorem assures that the above algorithm indeed gives a local solution of the Cauchy problem.

Theorem 4.5. *Consider a Cauchy problem for the PDE (4.1) with Cauchy data* $u_0(\eta)$ *prescribed on a curve* γ *given by (1.6), where* I *is an open interval, say for* $0 < \eta < 1$. *Let*
(i) the $F(x, y, u, p, q) \in \mathcal{C}^2(D_3)$, *where* D_3 *is a domain in* (x, y, u, p, q)*-space,*
(ii) the functions $x_0(\eta), y_0(\eta), u_0(\eta) \in \mathcal{C}^2(I)$,
(iii) $p_0(\eta)$ *and* $q_0(\eta)$ *be two functions satisfying the equations (4.8) and (4.9) such that they are* $\mathcal{C}^1(I)$ *and the set* $\{x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)\} \in D_3$ *for* $\eta \in I$ *and*
(iv) the transversality condition

$$\frac{dx_0}{d\eta}F_q(x_0, y_0, u_0, p_0, q_0) - \frac{dy_0}{d\eta}F_p(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)) \neq 0\ , \eta \in I \tag{4.12}$$

is satisfied.
Then we can find a domain D *in* (x, y)*-plane containing the datum curve* γ *and a unique solution of the Cauchy problem.*

A note on the proof of the above theorem: The algorithm mentioned above for construction of the solution is quite similar to that for a quasilinear equation except that we now have to integrate a system of five ordinary differential equations. The proof of the Theorem 4.5. is also similar to that of the Theorem 3.4. but in addition we need to prove that p and q obtained by solving the Charpit equations are indeed the first partial derivatives of the function $u(x, y)$. This makes the proof of the Theorem 4.5. very long compared to that of the Theorem 3.4., see [3, 10].

Characteristic Cauchy problem: Important point for the existence and uniqueness of the Cauchy problem is that the datum curve γ is no where tangential to a characteristic curve.

If γ is a characteristic curve, the data $u_0(\eta)$ is to be restricted (i.e., u_0, p_0 and q_0 satisfy the last three of the Charpit equations with σ replaced by η) and when this restriction is imposed, the solution of the Cauchy problem is non-unique, in fact infinity of solutions exist.

Example 4.6. Consider a wavefront moving into a uniform two-dimensional medium with a constant normal velocity c. Let the successive positions of the wavefront be denoted by an equation $u(x, y) = ct$, where t is the time. Since the velocity of propagation c is given by $c = -\frac{\phi_t}{({\phi_x}^2 + {\phi_y}^2)^{1/2}}$, where $\phi := ct - u(x, y) = 0$, the function $u(x, y)$ satisfies an eikonal equation

$$p^2 + q^2 = 1; \; p = u_x, q = u_y. \tag{4.13}$$

Let the initial position of the wavefront be given by

$$\alpha x + \beta y = 0, \quad \alpha, \beta = constants, \quad \alpha^2 + \beta^2 = 1. \tag{4.14}$$

We can formulate the problem of finding successive positions of the wavefront for $t > 0$ as a Cauchy problem.

Solve the PDE (4.13) subject to the Cauchy data at $\sigma = 0$

$$x_0 = \beta\eta, \; y_0 = -\alpha\eta, \; u_0 = 0. \tag{4.15}$$

Solution is quite easy. First we find out the values of p_0 and q_0 from (4.8)-(4.9), i.e,

$$p_0^2 + q_0^2 = 1, \; \beta p_0 - \alpha q_0 = 0$$

which give two sets of values at $\sigma = 0$.

$$p_0 = \pm\alpha, \; q_0 = \pm\beta. \tag{4.16}$$

The Charpit's equations of (4.13) are

$$\frac{dx}{d\sigma} = 2p, \; \frac{dy}{d\sigma} = 2q, \tag{4.17}$$

$$\frac{du}{d\sigma} = 2\left(p^2 + q^2\right), \tag{4.18}$$

$$\frac{dp}{d\sigma} = 0, \; \frac{dq}{d\sigma} = 0. \tag{4.19}$$

Solution of (4.17)-(4.19) with initial data (4.15)-(4.16) at $\sigma = 0$ gives

$$x = \pm 2\alpha\sigma + \beta\eta, \; y = \pm 2\beta\sigma - \alpha\eta, \tag{4.20}$$

$$u = 2\sigma, \; p = \pm\alpha, \; q = \pm\beta. \tag{4.21}$$

Solving σ in terms of x and y from (4.20), we get

$$\sigma = \pm\frac{1}{2}(\alpha x + \beta y).$$

From (4.21) we have $u = 2\sigma$ and hence the solutions of the Cauchy problem (4.13) and (4.15) are

$$u = \pm(\alpha x + \beta y). \tag{4.22}$$

There are two solutions of this Cauchy problem. Note that this does not violate the uniqueness theorem for a solution of the Cauchy problem for a first order nonlinear PDE. The two solutions correspond to the two sets of determinations of p_0 and q_0 in (4.15) leading actually to two Cauchy problems.

The two solutions have an important physical interpretation in terms of front propagation. The wavefronts (one forward propagating and another backward propagating) starting from the position (4.14) occupy the positions $u = ct$, i.e,

$$\alpha x + \beta y = \pm ct \tag{4.23}$$

which are at a (normal) distance $\pm ct$ from (4.14). The equation (4.13) governs both the forward and backward moving waves.

4.3 Some more results and general remarks

Theorem 4.7. *If $\phi(x, y)$ and $\psi(x, y)$ are two solutions of the nonlinear equation (4.1) in a domain D and they have common values of u_0, p_0 and q_0 at a point $(x_0, y_0) \in D$, then they have a common characteristic in D passing through (x_0, y_0) and that the values of u, p and q on this common characteristic are same for both solutions.*

Proof. As in the case of the proof of the Theorem 3.7. we use the uniqueness of the solution of an initial value problem for the Charpit's equations (4.3)-(4.6). Rest of the arguments are the same □

Remark 4.8. We have presented the theory of first order PDEs briefly. It is based on the existence of characteristics curves in the (x, y)-plane. Along each of these characteristics we derive a number of compatibility conditions, which are transport equations and which are sufficient to carry all necessary information from the datum curve in the Cauchy problem into a domain in which the solution is determined. In this sense every first order PDE is a hyperbolic equation[8].

Remark 4.9. We have omitted a special class of solutions known as complete integral, for which any standard text may be consulted. Every solution of the PDE (4.1) can

[8] A classification of equations into hyperbolic and other type of equations is done for a single higher order equation or for a system of first order equations. The hyperbolicity of equations is due to the fact that the equation or the system has sufficient number of families of characteristic curves (or bicharacteristics for equations in more than two independent variables), which carry all necessary information from the datum curve (or surface for equations in more than two independent variables) to a point P not on the datum curve (or surface) to construct the solution of a Cauchy problem at the point P

be obtained from a complete integral. We can also solve a Cauchy problem with its help. Though complete integral plays an important role in physics, it is not important for further development of the theory of first order PDEs and conservation laws.

Remark 4.10. So far we have discussed only genuine solutions, which is valid only locally. There is a fairly complete theory of weak solutions of Hamilton-Jacobi equations, a particular case of the nonlinear equation (4.1). Generally the domain of validity of a weak solution with Cauchy data on the x-axis is at least half of the (x, y)-plane. Theory of a single conservation law, a first order equation, is particularly interesting not only from the point of view of theory but also from the point of view of applications [9, 10].

4.4 Problem set

1. Consider the partial differential equation

$$F \equiv u(p^2 + q^2) - 1 = 0.$$

   (i) Show that the general solution of the Charpit's equations is a four parameter family of strips represented by

$$x = x_0 + \frac{2}{3}u_0(2\sigma)^{\frac{3}{2}}\cos\theta,\ y = y_0 + \frac{2}{3}u_0(2\sigma)^{\frac{3}{2}}\sin\theta,$$
$$u = 2u_0\sigma,\ p = \frac{\cos\theta}{\sqrt{2\sigma}},\ q = \frac{\sin\theta}{\sqrt{2\sigma}}$$

   where x_0, y_0, u_0 and θ are the parameters.

   (ii) Find the three parameter family of all Monge strips.

   (iii) Show that the characteristic curves consist of all straight lines in the (x, y)-plane.

2. Solve the following Cauchy problems:

   (i) $\frac{1}{2}(p^2 + q^2) = u$ with Cauchy data prescribed on the circle $x^2 + y^2 = 1$ by

$$u(\cos\theta, \sin\theta) = 1,\ \ 0 \le \theta \le 2\pi$$

   (ii) $p^2 + q^2 + \left(p - \frac{1}{2}x\right)\left(q - \frac{1}{2}y\right) - u = 0$ with Cauchy data prescribed on the x-axis by

$$u(x, 0) = 0$$

   (iii) $2pq - u = 0$ with Cauchy data prescribed on the y-axis by

$$u(0, y) = \frac{1}{2}y^2$$

   (iv) $2p^2x + qy - u = 0$ with Cauchy data on x-axis

$$u(x, 1) = -\frac{1}{2}x.$$

5 Existence and propagation of singularities on a characteristic curve

Certain type of singularities in a solution of a hyperbolic equation in general and a first order PDE in particular exist on a characteristic curve.

For a linear first order PDE, all types of singularities in the solution propagate along characteristic curves. However, the discussion would require using concept of a weak solution in a more general sense than what we use here. In this section we shall restrict ourselves to singularities which are discontinuities with finite jumps in the first derivatives for linear and quasilinear equations and in the second derivatives for nonlinear equations.

We remark here that the coefficients of all transport equations (5.18), (5.26) and (5.27) are C^1 functions of σ. Hence their solutions exist locally in a neighbourhood of $\sigma = 0$.

5.1 Kinematical conditions on a curve of discontinuity

Consider a smooth curve $\Omega : \varphi(x, y) = 0$ in a domain D in (x, y)-plane such that Ω divides D into two subdomains D_- and D_+ on two sides of Ω, i.e., $D = D_- \cup \Omega \cup D_+$.

Remark 5.1. The operators $\varphi_x \frac{\partial}{\partial x} + \varphi_y \frac{\partial}{\partial y}$ and $\varphi_y \frac{\partial}{\partial x} - \varphi_x \frac{\partial}{\partial y}$ at a point on Ω represent derivatives in normal and tangent directions with respect to the curve Ω.

Definition 5.2. Let D^m_Ω, where m is a non-negative integer, be the space of functions $u : D \to \mathbb{R}$ belonging to $\mathcal{C}^m(D \backslash \Omega)$ such that the limiting values of the partial derivatives of u of order up to m, as we approach Ω from the two sides, exist.

Definition 5.3. For a smooth function f, except for a finite jump discontinuity on Ω, and with limiting values f_- and f_+ as we approach Ω from D_- and D_+ respectively, i.e., for $f \in D^0_\Omega$, the jump $[f]$ across Ω is defined by

$$[f] = f_- - f_+. \tag{5.1}$$

Remark 5.4. If $u \in D^1_\Omega \cap \mathcal{C}(D)$, jump in the tangential derivative of u on Ω is zero, i.e.,

$$[\varphi_y u_x - \varphi_x u_y] = 0 \tag{5.2}$$

but, if there is a nonzero jump in any first derivative of u on Ω, the jump in the normal derivative is nonzero, i.e.,

$$[\varphi_x u_x + \varphi_y u_y] \neq 0. \tag{5.3}$$

A transversal derivative with respect to the curve Ω, $\frac{\partial}{\partial\zeta}$ say, i.e., a differentiation in a direction other than the tangential direction is a linear combination of two operators in remark (5.1). Thus

$$\frac{\partial}{\partial\zeta} = \lambda\left(\varphi_x\frac{\partial}{\partial x} + \varphi_y\frac{\partial}{\partial y}\right) + \mu\left(\varphi_y\frac{\partial}{\partial x} - \varphi_x\frac{\partial}{\partial y}\right), \quad \lambda \neq 0. \tag{5.4}$$

Hence (5.3) implies that the jump in the derivative of u in a *transversal* direction, namely

$$\left[\frac{\partial u}{\partial\zeta}\right] \neq 0. \tag{5.5}$$

Problem

Examine discontinuities in first order derivatives of $u = |x - t|$.

Theorem 5.5. *If $u \in D^1_\Omega \cap \mathcal{C}(D)$, then*

$$[u_x] = (\varphi_x|_\Omega)\omega_1, [u_y] = (\varphi_y|_\Omega)\omega_1, \tag{5.6}$$

where $\omega_1 : \Omega \to \mathbb{R}$ is a measure of the strength of the discontinuity in the first derivatives of u.

Proof. This follows from (5.2). □

Theorem 5.6. *If $u \in D^2_\Omega \cap \mathcal{C}^1(D)$, then*

$$[u_{xx}] = (\varphi_x|_\Omega)^2\omega_2, [u_{xy}] = (\varphi_x|_\Omega\varphi_y|_\Omega)\,\omega_2, [u_{yy}] = (\varphi_y|_\Omega)^2\omega_2, \tag{5.7}$$

where $\omega_2 : \Omega \to \mathbb{R}$ is a measure of the strength of discontinuities in the second derivatives of u.

Proof. Hint: The tangential derivatives of u_x and u_y on Ω are continuous. □

5.2 Linear and semilinear equations

Consider the PDE (2.9), namely

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \tag{5.8}$$

and a generalised or weak solution u of this equation in a domain D such that $u \in D^1_\Omega \cap \mathcal{C}(D)$. Note we have not defined a generalised solution but for this particular equation let us accept that $u \in D^1_\Omega \cap \mathcal{C}(D)$ is a generalised solution if u is a genuine solution in D_1 and D_2. Unlike the classical solution which must be a $\mathcal{C}^1(D)$ function, the first derivatives of a generalised solution may suffer finite jump across $\Omega : \varphi(x,y) = 0$.

Remark 5.7. Consider a generalised solution of (5.8). If ω_1, appearing in (5.6), vanishes then u is a genuine solution in D.

Remark 5.8. If $u \in D^2_\Omega \cap \mathcal{C}(D)$ and u satisfies (5.8) in D_- and D_+, then also u is a generalised solution.

Now we prove a theorem.

Theorem 5.9. *If Ω is a curve of discontinuity of the first order derivatives of the generalised solution $u \in D^1_\Omega \cap \mathcal{C}(D)$ then Ω is a characteristic curve.*

Proof. Let us make a transformation of independent variables (x, y) to (φ, ψ), where ψ is another smooth function such that $J := \partial(\varphi, \psi)/\partial(x, y) \neq 0$. The equation (5.8) transforms to

$$(a\varphi_x + b\varphi_y)u_\varphi + (a\psi_x + b\psi_y)u_\psi = c. \tag{5.9}$$

Since $(\partial/\partial\psi) = (1/J)\left(\varphi_y \frac{\partial}{\partial x} - \varphi_x \frac{\partial}{\partial y}\right)$, u_ψ is a derivative of u in the direction of tangent to Ω and hence it is continuous on Ω. The coefficients of u_φ, u_ψ and the function c are also continuous on Ω. Taking jump of the equation (5.9) across Ω, we get

$$(a\varphi_x + b\varphi_y)|_\Omega [u_\varphi] = 0. \tag{5.10}$$

Since $(\partial/\partial\varphi) = (1/J)\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y}\right)$, u_φ is a derivative in the direction of tangent to the curve ψ =constant and hence a transversal derivative of u with respect to the curve Ω. Hence $[u_\varphi] \neq 0$ and from the equation (5.10) we get

$$a\varphi_x + b\varphi_y = 0 \text{ on } \Omega. \tag{5.11}$$

(5.11) implies that the tangent direction of $\Omega : \varphi(x, y) = 0$ satisfies $\frac{dy}{dx} = \frac{b}{a}$. Hence the curve Ω is a characteristic curve. □

Example 5.10.

$$g(x, y) = \begin{cases} x + y, & x > 0 \\ (x + y)^2, & x < 0 \end{cases} \tag{5.12}$$

is a solution of

$$u_x - u_y = 0 \tag{5.13}$$

in any domain in (x, y)-plane which does not intersect the y-axis. But note that this function is discontinuous on on a non-characteristic line $x = 0$ and hence is not a generalised solution in $\mathbb{R}^2$. A generalised solution of (5.13) is

$$h(x, y) = \begin{cases} x + y, & x + y > 0 \\ (x + y)^2, & x + y < 0. \end{cases} \tag{5.14}$$

Note that the first derivatives of h are not continuous on Ω, which is a characteristic curve. Here, $[h_x] = -1 = [h_y]$ on Ω.

For a generalised solution u, the curve Ω of discontinuity of the first derivatives of u is a characteristic curve, which is given by the equation

$$\frac{dx}{d\sigma} = a(x, y), \ \frac{dy}{d\sigma} = b(x, y) \tag{5.15}$$

Since both vectors (a, b) and $(\varphi_y, -\varphi_x)$ are in tangent direction of Ω, we can replace the relation (5.6) by

$$[u_x] = -b|_\Omega W_1, \ [u_y] = a|_\Omega W_1 \tag{5.16}$$

where $W_1 = (\frac{\phi_y}{a})|_\Omega \omega_1$, a function of σ, is a new measure of the strength of the discontinuity in the first derivatives of u.

Theorem 5.11. *If the strength of the discontinuity in the first derivatives of a generalised solution $u \in D^2_\Omega \cap \mathcal{C}(D)$ of (5.8) is known at one point of the curve of discontinuity Ω, then it is uniquely determined at all points of Ω.*

Remark 5.12. In the proof of this theorem we need the continuity of the second derivatives of u in D_1 and D_2 in order to use $u_{yx} = u_{xy}$ in both subdomains.

Proof. Differentiating (5.8) with respect to x and then taking jump across Ω, we get

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)[u_x] = (c_u - a_x)[u_x] - b_x[u_y]. \tag{5.17}$$

Substituting (5.16) in this we get

$$b\frac{dW_1}{d\sigma} = \left\{b(c_u - a_x) - \frac{db}{d\sigma} + ab_x\right\} W_1 \quad \text{on } \Omega \tag{5.18}$$

where $\frac{d}{d\sigma} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and a, b, a_x, b_x, c_x and c_u are functions of σ on Ω. This equation contains the variable u. The variation of u along the characteristic curve is given by (2.12).

Let us assume that the amplitude W_{10} of the discontinuity in derivative is known at a point (x_0, y_0), where the value of the solution is u_0. Equations (5.15) give the characteristic Ω through (x_0, y_0). Then equation (2.12) gives the value of the solution on Ω. Finally the transport equation (5.18) gives the value of W_1 on Ω and shows that if W_1 is known at some point of the characteristic curve Ω, it is determined at all points of Ω. All these results are local. □

Remark 5.13. For a linear PDE, the characteristic curves carry the values of $u, [u_x]$ and $[u_y]$ throughout the domain where characteristics are defined. For a semilinear equation, since the compatibility condition $\frac{du}{d\sigma} = c(x, y, u)$ is nonlinear the solution may blow up even if characteristics are well defined. Then, of course, u_x and u_y are not defined. It is simple to verify this statement in 4(iii) in Problem set 2.3.

5.3 Quasilinear equations

Consider the PDE (3.1), namely

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{5.19}$$

Take a generalised solution $u(x, y) \in D^1_\Omega \cap \mathcal{C}(D)$. Since $u \in \mathcal{C}(D)$, for this known solution, the functions

$$A(x, y) = a(x, y, u(x, y)),\ B(x, y) = b(x, y, u(x, y)),\ C(x, y) = c(x, y, u(x, y)) \tag{5.20}$$

are continuous functions in D. As in the case of semilinear equations, we can prove the following theorem.

Theorem 5.14. *The curve Ω of discontinuity of the first derivatives of a generalised solution $u \in D^1_\Omega \cap \mathcal{C}(D)$ of (5.19) is a characteristic curve given by*

$$\frac{dx}{d\sigma} = a(x, y, u(x, y)),\ \frac{dy}{d\sigma} = b(x, y, u(x, y)) \tag{5.21}$$

Proof. The proof is similar to that of the theorem (5.11). □

We could proceed to derive the transport equation for the measure W_1 of the strength of the discontinuity satisfying (5.16) but we leave it as a simple exercise. Instead we derive transport equation for $[u_x]$ assuming (as explained in (5.8) that $u \in D^2_\Omega \cap \mathcal{C}(D)$. We differentiate (5.19) with respect to x to get

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) u_x = c_x + c_u u_x - a_x u_x - a_u(u_x)^2 - b_x u_y - b_u(u_x u_y). \tag{5.22}$$

Taking jump of this equation across Ω, we get

$$\frac{d[u_x]}{d\sigma} = (c_u - a_x)[u_x] - b_x[u_y] - a_u[(u_x)^2] - b_u[u_x u_y]. \tag{5.23}$$

Now two new terms appear, which we deal as follow

$$[(u_x)^2] = u^2_{x-} - u^2_{x+} = (2u_{x+} + [u_x])[u_x], \tag{5.24}$$

$$[u_x u_y] = u_{x-}u_{y-} - u_{x+}u_{y+} = [u_x][u_y] + u_{x+}[u_y] + u_{y+}[u_x]. \tag{5.25}$$

Substituting (5.24) and (5.25) in (5.23) and using $[u_y] = -\frac{a}{b}|_\Omega[u_x]$, we get

$$\frac{d[u_x]}{d\sigma} = \left\{c_u + \frac{ab_x - ba_x}{b} - b_u\frac{bu_{y+} - au_{x+}}{b} - 2a_u u_{x+}\right\}[u_x] + \frac{ab_u - ba_u}{b}[u_x]^2 \tag{5.26}$$

which is the required transport equation. This is again an ODE for variation of the jump $[u_x]$ but involves the variables u, u_{x+} and u_{y+}. Only one of u_{x+} and u_{y+} is independent since they satisfy the PDE (5.19). The equation for u is the compatibility condition (3.4). We can easily derive the equations for u_{x+} from (5.22) in the form

$$\frac{du_{x+}}{d\sigma} = c_x + (c_u - a_x)u_{x+} - a_u(u_{x+})^2 - b_x u_{y+} - b_u(u_{x+}u_{y+}). \tag{5.27}$$

In addition to the value u_0 and strength $[u_x]_0$ of the discontinuity at the point (x_0, y_0), we also need value of $(u_{x+})_0$. The characteristic equations (3.3), the compatibility condition (3.4) give the characteristic curve Ω and the value of u on it. The transport equation (5.27) gives u_{x+} on Ω. Finally the transport equation (5.26) gives the strength of the discontinuity $[u_x]$ on Ω.

Therefore, if the strength $[u_x]_0$ of the initial discontinuity is given at a point (x_0, y_0), it is uniquely determined at all points of the characteristic. As in case of a semilinear equation, the the theorem (5.10) remains true also for a quasilinear equation.

Since the equation (5.26) is nonlinear, the initial discontinuity $[u_x]$ of finite value may tend to infinity in finite time t_c, say, and at that time the continuity of the solution $u \in \mathcal{C}(D)$ breaks down. The limiting values of the solution u from two sides of Ω do remain finite and a *shock* appears in the solution at t_c and at the place, where $[u_x] \to \infty$. We do not wish to spend much time on the equation (5.26) but an example of its application to Burgers' equation is given in [8], see also [9]. The phenomena of break down of continuity of the solution and appearance of a shock is very important in the theory of hyperbolic conservation laws with *genuine nonlinearity.* It now forms an important part of a basic course in PDE.

5.4 Nonlinear PDE

For a nonlinear PDE (4.1), the expressions F_p and F_q on the right hand sides of the characteristic equations (4.3) contain the partial derivatives u_x and u_y. Hence a curve of discontinuity in u_x and u_y can not coincide with the characteristic curves on both sides of it. Analysis of such curves of discontinuities will be for more complicated. Hence we consider a genuine solution $u \in D^2_\Omega \cap C^1(D)$ of (4.1), for which we assume that the second order derivatives are discontinuous, i.e, ω_2 in (5.7) is nonzero. We can easily prove that Ω is a characteristic of (4.1).

For the derivation of the transport equation of the discontinuities in second order derivatives, we shall have to take $u \in D^3_\Omega \cap \mathcal{C}^1(D)$ (as explained in (5.8)) and differentiate(4.4) or (4.5) in the characteristic direction, use (5.7) and then derive the transport equation for ω_2 or one of the second derivatives. The resultant equation will be quite long. It will be nice to derive this transport equation for a particular equation such as $u_x^2 + u_y + u = 0$. We leave it as an exercise to the reader to deduce

$$\frac{d[u_{xx}]}{dy} = -(1 + 4u_{xx+})[u_{xx}] - 2[u_{xx}]^2 \tag{5.28}$$

along the characteristic curves given by the Charpit's equations

$$\frac{dx}{dy} = 2u_x, \ \frac{du}{dy} = 2(u_x)^2 + u_y, \ \frac{du_x}{dy} = -u_x, \ \frac{du_y}{dy} = -u_y. \tag{5.29}$$

The evolution of u_{xx+} along the characteristic curve is given by

$$\frac{du_{xx+}}{dy} = -u_{xx+} - 2u_{xx+}^2. \tag{5.30}$$

For a nonlinear equation in general, and for the equation $u_x^2 + u_y + u = 0$ in particular, there is a relation between u, u_x and u_y and hence only two, say u and u_x, can be prescribed at any point (x_0, y_0). Now the four equations in (5.29) are to be solved simultaneously with initial data $u_0, (u_x)_0, (u_y)_0$ at a point (x_0, y_0). This will give the characteristic curve Ω through (x_0, y_0) and the values of u, u_x and u_y on it. Then, if the strength $[u_{xx+}]_0$ of the discontinuity is prescribed at (x_0, y_0), we can solve (5.30) to get the value of $[u_{xx+}]$ on Ω.

Therefore, if the strength $[u_{xx}]_0$, of the initial discontinuity is given at a point (x_0, y_0), it exist along the characteristic curve through the point (x_0, y_0) and is uniquely determined at all points of the characteristics in a neibourhood of (x_0, y_0).

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References

[1] R. Courant and D. Hilbert. *Methods of Mathematical Physics*, vol 2: Partial Differential Equations. Interscience Publishers, 1962.

[2] S. S. Demidov. The Study of Partial Differential Equations of the First Order in the 18 th and 19 th Centuries. Archive for History of Exact Sceinces, 1982, **26**, 325-350.

[3] L. C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics, Vol 19, American Mathematical Society, 1999.

[4] P. R. Garabedian. *Partial Differential Equations*. John Wiley and Sons, 1964.

[5] E. Goursat. *A Course in Mathematical Analysis, Vol II, Part Two: Partial Differential Equations*. Dover Publication of the original book published by Ginn and Company, 1917.

[6] F. John. *Partial Differential Equations.* Springer-Verlag, 1982.

[7] L. Perko. *Differential Equations and Dynamical Systems.* Springer, 2001.

[8] Phoolan Prasad. A theory of first order PDE through propagation of discontinuities. Ramanujan Mathematical Society News Letter, 2000, **10**, 89-103.

[9] Phoolan Prasad. *Nonlinear Hyperbolic Waves in Multi-dimensions.* Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, **121**, 2001.

[10] Phoolan Prasad and Renuka Ravindran. *Partial Differential Equations.* Wiley Eastern Ltd, 1985.

[11] I. N. Sneddon. *Elements of Partial Differential Equations.*, McGraw-Hill,1957.