Focusing of weak acoustic shock waves at a caustic cusp

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Abstract

This paper investigates theoretically the focusing of weak acoustic shock waves at a caustic cusp. Near the cusp, a diffraction boundary layer is introduced, the characteristic length-scales of which are determined with the help of Pearcey function which governs the field in linear acoustics. With the adequate variables, the nonlinear wave equation is then shown to reduce to the KZ equation. The proper matching to the geometrical acoustics approximation provides the complete boundary conditions, so that the problem is numerically tractable. A similitude is found for an incoming step wave, and the linear approximate solution is obtained in the general case. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Weak shock waves arise in several situations, either naturally or through man activity, such as thunder, sonic boom of supersonic aircraft, hydraulic jumps, water waves breaking or high intensity focused ultrasound (HIFU) used for medical therapy. Focusing of shock waves is likely to happen in all of these situations. In most cases, wave focusing occurs at surfaces called caustics. Caustics can be of several types, classified according to the theory of catastrophes. The two simplest cases and the most likely to occur, are the smooth caustic, and the cusp caustic, a surface having an “arête”. Perfect focus at one point is unlikely to occur, except for on purpose applications such as HIFU.

Caustics are both physical objects — regions of wave amplification — and geometrical ones — surfaces where the approximate ray theory gets singular. Ray theory neglects diffraction, an assumption that becomes invalid near caustics. To describe the wave field there, a more elaborate theory, the geometrical theory of diffraction, is required. It introduces a diffraction boundary layer around the caustic where first-order diffraction effects are recovered, and then matches it to ray theory far away from the caustic. This process has been first elaborated for smooth or cusp caustics and continuous waves by Buchal and Keller [1] and Ludwig [2]. In particular, the wavefield can always be described by generic functions (Airy function for smooth caustics, Pearcey function for cusp caustics), which are intimately related to the local behavior of the phase function as classified by the theory of catastrophes. This has been widely studied, and nicely reviewed by Marston [3].

However, for weak shock waves, this process is not sufficient to predict the field amplitude. Indeed, shock wave focusing is fundamentally a nonlinear process, though this feature has been controversial for a long time. It seems now widely accepted that, for weak shock waves, the linear mechanism of wavefront folding remains, but nonlinearities must nonetheless be taken into account to prevent the formation of sharp peaks, an effect observed in the experiments of Sturtevant and Kulkarny [4].

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From a theoretical point of view, weak shock focusing at a smooth caustic has been first elucidated during the 1960s, motivated by the concern about the focusing of sonic boom induced by maneuvering supersonic aircraft (superboom). Using considerable mathematics, Guiraud [5] elaborated a consistent theory including both diffraction and nonlinear effects at first order and leading to the nonlinear Tricomi equation. He also derived a nonlinear similitude law relating the maximum overpressure at the caustic to the power $4 \times$ of the incoming wave amplitude. These results were confirmed by Hayes [6], who relied on rather intuitive arguments, and Pechuzal and Kevorkian [7], who used explicitly the formalism of matched asymptotic expansions in the case of focusing induced by an airplane flying at constant speed in a two-dimensional atmosphere with a weak wind gradient. In a review report, Coulouvrat [8] extended the previous results to a three-dimensional heterogeneous medium. Hunter and Keller [9] showed that the nonlinear Tricomi equation occurs for the general case of weakly nonlinear wave solutions of a system of hyperbolic equations. Mentioning only the last of these references, Rosales and Tabak [10] again recovered the Tricomi equation and Guiraud’s similitude for a two-dimensional homogeneous atmosphere. They also pointed out a new possible singularity of the solutions of Tricomi equation occurring near triple shocks intersection, a fact that seems to be supported by very fine numerical simulations [11].

Seebass [12] and Gill and Seebass [13] provided an approximate analytical solution for an incoming step wave by transforming the nonlinear Tricomi equation into the linear one through the hodograph transformation. Keeping the untransformed linear boundary condition, the solution can be found analytically, and then is returned into the physical plane. This process leads to the apparition of either multivalued or empty parts in the wave profiles, unphysical occurrences which are overcome by introducing shock waves or expansion fans. This rather complex procedure can be achieved satisfactorily only for small amplitude waves, but then it can be extended to higher amplitude waves on the basis of Guiraud’s similitude. This technique has been extended by Fung [14] to a broader class of incoming signals, and used for simulations of sonic boom focusing [15]. Comparisons with peak overpressures measured for sonic boom focusing due to maneuvering aircraft [16], or for low supersonic bullets fired in a stratified mixture of air and carbon dioxide [17] show reasonable agreement with approximate theory [18]. Recently, a comprehensive experimental study [19] showed that the focus position could be predicted with a good accuracy provided nonlinear effects are included, while the maximum pressure could be reasonably well estimated.

Direct numerical simulations of the nonlinear Tricomi equation for an incoming step wave have been performed using first a shock-capturing [20], then a shock-fitting [21] finite-difference algorithm. Recently, numerical simulations of a general shock wave profile such as an “N” wave, based on a pseudospectral algorithm, have been performed [22], and compared favorably with Guiraud’s similitude.

In comparison, the cusp caustic has been the object of relatively few investigations. Pierce [23] used heuristic arguments to derive a similitude law, and made a tentative comparison with experimental results [16]. The method of matched asymptotic expansion was used by Cramer and Seebass [24], who derived an evolution equation for a step wave incoming at a cusp caustic. Cramer [25] could then obtain a similitude law and generalize his results to the three-dimensional or axisymmetric cases [26]. In the present paper, we extend Cramer’s results to the case of a general waveform (Section 4). We complete in a simplified way the matching to geometrical acoustics, so that all the boundary conditions are properly specified and the problem now appears to be numerically tractable (Section 5). We also show how the scaling of the variables is related to the intrinsic Pearcey function (Section 3), a feature supporting the asymptotic analysis. We then partly confirm Pierce’s similitude (Section 6), and calculate the linear approximate solution (Section 7). Finally, the focusing of a weak shock wave at either a smooth or a cusp caustic are compared.

2. Caustic geometry

Let us consider a wave propagating in two dimensions in an inviscid homogeneous fluid with ambient sound speed $c_0$. Initially, the wave is specified at a concave wavefront, whose equation is

$$z = f(x).$$
If the radius of curvature of this wavefront has a minimum \( R_0 \), it is well known that a cusp caustic will appear. The origin \( O \) is chosen at the point of minimum radius of curvature. The \( Oz \) axis is normal to the wavefront, directed towards the propagation direction, and the \( O\alpha \) axis is tangential to the wavefront (Fig. 1).

According to Huygens’s principle, the acoustic pressure \( p \) of a continuous signal of frequency \( \omega/2\pi \) can be expressed at an observation point \( P(x, z) \) as a sum of cylindrically diverging waves emanating from the wavefront. Provided the observation point is far enough from the initial wavefront, the Hankel function can be replaced by its farfield asymptotic expression, so that

\[
p(x, z, t) = e^{-i\omega t} \int_{-\infty}^{+\infty} r^{-1/2} A(\alpha) e^{ikr} d\alpha,
\]

where \( k = \omega/c_0 \) is the wavenumber, \( A(\alpha) \) a smooth amplitude function, \( \alpha \) the \( x \)-coordinate of a current point \( M \) along the wavefront, and \( r \) the distance between the observation point \( P \) and the current point \( M \) (Fig. 1):

\[
r = \sqrt{(x - \alpha)^2 + (z - f(\alpha))^2}.
\]

Fermat’s principle states that among all lines connecting the observation point \( P \) and a wavefront point \( M \), the acoustic rays are those minimizing the distance \( MP \):

\[
\frac{\partial r}{\partial \alpha} = 0 \Leftrightarrow x + f'(\alpha)z - \alpha - f(\alpha)f'(\alpha) = 0,
\]

so that rays are the straight lines normal to the wavefront. Each individual ray can thus be identified by a single “ray-coordinate” \( \alpha \), which is the \( x \)-coordinate of its intersection with the initial wavefront.
Caustics are the locus of points where the first and second derivatives of the phase function simultaneously vanish (equivalently, caustics can be viewed as the locus of intersection points of infinitely adjacent rays). The caustic equation is given under parametric form by

\[ \frac{\partial^2 r}{\partial \alpha^2} = 0 \iff x + f'(\alpha)z - \alpha - f(\alpha) f'(\alpha) = 0, \quad \frac{\partial^2 r}{\partial \alpha^2} = 0 \iff f''(\alpha)z - 1 - f(\alpha) f''(\alpha) - f''(\alpha) = 0. \]  

(5)

Let us now study the caustic geometry near the ray emanating from the origin. We simultaneously have \( f(0) = 0 \) (choice of the origin), \( f'(0) = 0 \) (choice of the axis orientation), \( f''(0) = 0 \) (minimal radius of curvature) and \( f''(0) = 1/R_0 > 0 \). The intersection point \( C \) between the origin ray and the caustic has coordinates \((0, R_0)\). A point \( Q \) on the caustic can either be identified by its Cartesian coordinates \((x, z = R_0 + \delta)\) or by the ray-coordinate \( \alpha \) determining the ray which is tangent to the caustic at this point. According to Eqs. (5), it is easily checked that

\[ \delta = \frac{\alpha^2}{2} \left( 3 f''(0) - \frac{f''(0)}{f''(0)} \right) + O(\alpha^3), \quad x = -\frac{\alpha^3}{3} f''(0) \left( 3 f''(0) - \frac{f''(0)}{f''(0)} \right) + O(\alpha^4). \]

(6)

The quantity \( R_0'' = 3 f''(0) - (f''(0)/f''(0)) \) is the second derivative at the origin of the radius of curvature of the wavefront with respect to \( \alpha \). It is a positive quantity, as the radius of curvature is minimum at the origin. Consequently, the caustic equation can be locally written around point \( C \) [27]:

\[ x = \pm \frac{\delta^{3/2}}{\alpha^{1/2}}, \]

(7)

where the distance \( a = \frac{9}{8} R_0^2 R_0'' \) is the sole parameter determining the local geometry of the caustic.

3. Pearcey function and the characteristic length-scales of the diffraction boundary layer

In order to determine the characteristic length-scales governing the wavefield around the cusp, we study the phase function at an observation point near the caustic cusp. At high frequencies, the phase oscillations in Eq. (2) imply that the main contributions to the sound field around the cusp emanate from the portion of the wavefront near the origin \( O \). So an asymptotic expansion of the phase function can be obtained under the assumptions that: (i) values of \( \alpha \) are small and (ii) the coordinates \((x, z = R_0 + \delta)\) of the observation point \( P \) are such that \( x = O(\alpha^3) \) and \( \delta = O(\alpha^2) \) (point \( P \) is near the cusp \( C \), Eqs. (6)). A straightforward asymptotic expansion of the phase can then be deduced:

\[ r = R_0 + \delta - x R_0 - \frac{\delta}{2} \left( \frac{\alpha}{R_0} \right)^2 + \frac{a}{27} \left( \frac{\alpha}{R_0} \right)^4 + O(\alpha^5). \]

(8)

Substituting this asymptotic expression into Eq. (2) leads to the asymptotic expression of the pressure field near the cusp [3]:

\[ p(x, z, t) \approx \frac{A(0)}{R_0^{1/2}} \exp[\imath(kz - \omega t)] \int_{-\infty}^{+\infty} \exp \left[ -\imath k \frac{\alpha}{R_0} \frac{\delta}{2} \left( \frac{\alpha}{R_0} \right)^2 + \imath k \frac{a}{27} \left( \frac{\alpha}{R_0} \right)^4 \right] d\alpha. \]

(9)

Introducing the variable \( \tilde{\alpha} = (\alpha/R_0)(ka/27)^{1/4} \) yields an analytical expression of the pressure field in terms of the Pearcey function [28] \( P(\tilde{x}, \tilde{z}) = \int_{-\infty}^{+\infty} \exp[\imath x \tilde{\alpha} + \imath z \tilde{\alpha}^2 + \imath a^4] d\tilde{\alpha} \) under the form

\[ p(x, z, t) \approx Cte \exp[\imath (k \delta - \omega t)] P(\tilde{x}, -\tilde{z}). \]

(10)
The dimensionless variables $\tilde{x}$ and $\tilde{z}$ are related to the physical ones by two characteristic diffraction length-scales $L_z$ along, and $L_x$ transversely to, the propagation axis $O_z$:

$$\tilde{z} = \frac{\delta}{L_z} = \frac{27 \delta}{2\varepsilon a} \quad \text{with} \quad L_z = \left( \frac{4a}{27k} \right)^{1/2}$$

$$\tilde{x} = \frac{x}{L_x} = \frac{27 x}{\varepsilon^{3/2} a} \quad \text{with} \quad L_x = \left( \frac{a}{27k^3} \right)^{1/4}.$$  \hfill (11)

It is noteworthy that the longitudinal diffraction length varies as power $-\frac{1}{2}$ of the frequency, while the transverse one varies as power $-\frac{3}{4}$. We recall that the characteristic thickness of the acoustic boundary layer around a smooth caustic depends on power $-\frac{2}{3}$ of frequency [1]. The diffraction parameter

$$\varepsilon = \left( \frac{27}{ka} \right)^{1/2}$$  \hfill (12)

measures the magnitude order of diffraction effects near the caustic. At sufficiently high frequencies, the wavelength is small compared to the caustic geometrical parameter $a$, so that this parameter is small. For sonic boom in air, the typical frequency is about 10 Hz, and the geometrical parameter can be estimated to be of the order of the radius of curvature of acoustic rays, i.e. 100 km. This yields a diffraction parameter of about 0.04, a longitudinal distance of about 300 m, and a transverse one of about 30 m.

With these dimensionless variables, the caustic equation (7) reduces to the universal form

$$\tilde{x}^2 = \frac{8}{\varepsilon^2} \tilde{z}^3.$$  \hfill (13)

4. The KZ equation near the caustic cusp

In an inviscid fluid, the propagation of weakly nonlinear waves is governed by the Kuznetsov equation [29] for the potential $\Phi$:

$$\frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \Delta \Phi = \frac{\partial}{\partial t} \left[ \frac{1}{c_0^2} \frac{B}{2A} \left( \frac{\partial \Phi}{\partial t} \right)^2 + (\nabla \Phi)^2 \right]$$ \hfill (14)

with $B/2A$ the nonlinearity parameter ($\beta = 1 + B/2A$). This equation is exact up to quadratic nonlinear terms.

According to Eq. (10), we expect the sound field near the cusp (the inner expansion) to depend on the dimensionless retarded time $\tilde{t} = \omega (t - z/c_0)$ describing the axial wave propagation over the wavelength-scale, and on the two variables $\tilde{x}$ and $\tilde{z}$ associated to diffraction. When nonlinear effects are taken into account, the sound field will not be monochromatic anymore, so that here $\omega$ is simply a characteristic frequency. The dimensionless potential $\tilde{\Phi}$ is chosen by $\Phi(x, z, t) = U_0 c_0 \tilde{\Phi}(\tilde{x}, \tilde{z}, \tilde{t})/\omega$, with $U_0$ the velocity amplitude of the acoustic field. With this set of variables, the Kuznetsov equation becomes

$$\frac{\partial^2 \tilde{\Phi}}{\partial \tilde{t}^2} - \frac{\partial^2 \tilde{\Phi}}{\partial \tilde{z}^2} - \beta \varepsilon \frac{M \partial}{\partial \tilde{t}} \left[ \left( \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} \right)^2 \right] = \varepsilon \frac{M \partial^2 \tilde{\Phi}}{4 \partial \tilde{z}^2} + M \frac{\partial}{\partial \tilde{t}} \left[ \frac{(\partial \tilde{\Phi})^2}{\partial \tilde{x}^2} - \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} + \varepsilon \frac{(\partial \tilde{\Phi})^2}{4 \partial \tilde{z}^2} \right].$$  \hfill (15)

The acoustic Mach number $M = U_0/c_0$ is a small parameter measuring the nonlinear effects. For sonic booms, the Mach number is never much $> 10^{-3}$. As a consequence, the right-hand side of Eq. (15) can be neglected compared to the left-hand side. We thus recover Eq. (13) derived in [24]. Only the coefficients are slightly different: as these authors studied only step shocks, they did not introduce any frequency, but instead a length characteristic for the wavefront. The approximation remains formally valid as long as the conditions $M \gg \varepsilon^2$ and $\varepsilon \gg M^2$ are satisfied.
Apparently, the first condition is not in the sonic boom case. However, as will be seen below, nonlinear effects near the caustic cusp are mostly due to the amplification of incoming shock waves, for which the characteristic frequency is much higher than 10 Hz and the diffraction parameter is much smaller.

Introducing the (dimensionless) pressure $\tilde{p} = \partial \Phi / \partial \tilde{t}$, we immediately deduce that it is a solution of the Khokhlov–Zabolotskaya (KZ) equation [30]:

$$\frac{\partial^2 \tilde{p}}{\partial \tilde{t} \partial \tilde{z}} - \frac{\partial^2 \tilde{p}}{\partial \tilde{x}^2} = \mu \frac{\partial^2 (\tilde{p}^2)}{\partial \tilde{t}^2}. \tag{16}$$

The parameter $\mu$ is a comparative measurement of nonlinear to diffraction effects

$$\mu = \frac{\beta M}{\varepsilon} = \beta M \left( \frac{ka}{27} \right)^{1/2}. \tag{17}$$

The standard KZ equation governs the diffracted field of a nonlinear collimated sound beam radiated by an acoustic plane source along the $Oz$ axis. In this case, the parameter is defined by $\mu = \beta M d$, where $d$ is a characteristic dimension of the emitting source. We showed here that the KZ equation also governs the nonlinear sound field in the vicinity of the caustic cusp. The main differences between the two cases are: (i) the power $\frac{1}{4}$ appearing in the definition of the small diffraction parameter for a caustic cusp, and (ii) the boundary conditions that are to be satisfied as they will be defined now.

5. Boundary conditions: matching to geometrical acoustics

The boundary conditions that are to be satisfied by the pressure $\tilde{p}$ of solution of Eq. (16) (the inner expansion) are obtained by matching it, at large distances away from the cusp, to the geometrical acoustic approximation (the outer expansion) when approaching the cusp. In the standard way of matched asymptotic expansions, we must identify the outer limit of the inner expansion, to the inner limit of the outer expansion.

The geometrical acoustic approximation is obtained by applying the method of stationary phase to the integral solution, Eq. (2). As we are interested only in the solution around the cusp, the phase function can be again approximated by Eq. (8). Near the cusp, rays passing through point $P(x, \delta)$ are identified by the ray-coordinate $\alpha$ solution of the stationary phase condition

$$\frac{\partial r}{\partial \alpha} = 0 \rightarrow \frac{x}{R_0} - \frac{\delta \alpha}{R_0^2} + \frac{4a\alpha^3}{27R_0^4} + O(\alpha^4). \tag{18}$$

Introducing the dimensionless variable $\tilde{\alpha}$, Eq. (18) reduces to

$$4\tilde{\alpha}^3 - 2\tilde{\alpha} - \tilde{x} = 0. \tag{19}$$

Simple algebraic shows that if the observation point is located in region I “outside” the caustic cusp $x^2 > \frac{8}{27} \tilde{z}^3$, there exists only one single ray (one real solution of Eq. (19)). On the contrary, in region II “inside” the caustic cusp $x^2 < \frac{8}{27} \tilde{z}^3$, there are three different rays passing through any observation point (three real solutions of Eq. (19)). The caustic is the locus of points where two of the rays coincide and are tangent to the caustic, while the cusp $C$ is the only point where these three rays coincide (Fig. 2). Rays go through region I without having tangented the caustic, so that faraway from the cusp in region I, the acoustic field will not be affected by diffraction effects. On the contrary, in region II, one of the rays goes through the observation point after having tangented the caustic, and will necessarily be affected by diffraction effects. As a consequence, matching boundary conditions with geometrical acoustics must be specified in region I only. From Eq. (8), the dimensionless phase function reduces to

$$\omega \left( \frac{1 - r}{c_0} \right) = \tilde{\alpha} + \tilde{\alpha}^2 \tilde{z} - \tilde{\alpha}^4 + O(\tilde{\alpha}^5). \tag{20}$$
The method of stationary phase also gives the field amplitude as proportional to the inverse square root of the second derivative of the phase function with respect to parameter $\alpha$:

$$\frac{\partial^2 r}{\partial \alpha^2} = 0 = -\frac{\delta}{R_0^2} + \frac{4\alpha^2}{9R_0^4} + O(\alpha^3). \quad (21)$$

In terms of dimensionless variables, the amplitude of the geometrical acoustic varies as $|6\delta^2 - \bar{z}|^{-1/2}$. Consequently, in region I, the geometrical acoustic approximation near the caustic cusp can be written in the time domain under the form

$$\tilde{\rho}(\tilde{x}, \tilde{z}, \tilde{t}) = \frac{1}{\sqrt{|6\delta^2 - \bar{z}|}} F(\tilde{t} + \tilde{\alpha} \tilde{x} + \tilde{\alpha}^2 \bar{z} - \tilde{\alpha}^4), \quad (22)$$

where $F(\tilde{t})$ is the incoming signal waveform, and $\tilde{\alpha}(\tilde{x}, \tilde{z})$ the only real root of Eq. (19).

This achieves the mathematical formulation of the problem governing the pressure field $\tilde{\rho}(\tilde{x}, \tilde{z}, \tilde{t})$ around the caustic cusp: it must satisfy the KZ equation (16), and at large distances away from the cusp in region I, match the asymptotic expression of the geometrical acoustic approximation Eq. (22). In the time domain, we must prescribe either a vanishing field at large times for a pulse signal, or a periodic boundary conditions for a continuous signal.
Boundary condition (22) gives an indication about the numerical procedure to solve the problem. We introduce a rectangular domain of discretization \([x_{\text{min}}, x_{\text{max}}] \times [z_{\text{min}}, z_{\text{max}}]\) (dark gray rectangle in Fig. 3), where the KZ equation is to be solved. The solution is obtained by advancing plane by plane away from the leftmost side of the domain \(z = z_{\text{min}}\), using a standard algorithm such as the “Bergen code” [31]. The procedure is initialized according to the boundary condition (22). For each advancing step, the condition (22) is also prescribed on the domain boundaries \(x_{\text{D}} = x_{\text{min}}, x_{\text{D}} = x_{\text{max}}\). This is repeated until the rightmost side \(z = z_{\text{max}}\) of the domain is reached. The discretization domain must be carefully selected, in order that its boundaries \(z = z_{\text{D}} = z_{\text{D}} = z_{\text{min}}, x = x_{\text{min}}, x = x_{\text{max}}\) all lie in region I, and are sufficiently far away from the caustic.

The numerical problem can also be simplified by recognizing that the mathematical formulation, Eqs. (16) and (22) is symmetric in the \(N_{x}\)-coordinate, so that the problem needs to be solved only for positive \(N_{x}\) values, with the additional boundary condition \(\partial \bar{p} / \partial x (\bar{x} = 0) = 0\).

For fixed \(\bar{x}\) and \(\bar{\xi} \to -\infty, \bar{p}\) is inversely proportional to \(\sqrt{|\bar{x}|}\) (similarly, for fixed \(\bar{\xi}\) and \(\bar{x} \to +\infty, \bar{p}\) is proportional to \(\bar{x}^{-1/3}\)). We thus recover the matching conditions of Ref. [24]. However, this condition is insufficient for a numerical solution, as we also need the complete boundary condition (22) in the \(\bar{x}\) direction, when \(\bar{x}\) and \(\bar{\xi}\) are simultaneously large.

6. Similitude

If we introduce the new scaling

\[
\bar{p} = \mu^{-1/3} \tilde{p}, \quad \bar{t} = \mu^{4/3} \tilde{t}, \quad \bar{x} = \mu \tilde{x}, \quad \bar{\xi} = \mu^{2/3} \tilde{\xi}, \quad \bar{\alpha} = \mu^{1/3} \tilde{\alpha},
\]

Fig. 3. Domain for a numerical simulation.
it is easy to check that in terms of variables \( O \), the mathematical problem remains unchanged, except that: (i) parameter \( \mu \) is replaced by 1 in Eq. (16), and (ii) the phase function in Eq. (22) is multiplied by \( \mu^{4/3} \). If the incoming signal \( F(t) \) is a perfect step shock, it is invariant by any scaling of the phase function, so that the mathematical problem does not depend on any parameter. The solution is therefore unique, as outlined by Cramer [25]. Returning to dimensional variables, we deduce from this result that the pressure field varies as the power \( \frac{2}{3} \) of the Mach number, i.e. of the incoming wave amplitude. This result was already obtained by Pierce [23]; it is to be compared with Guiraud’s similitude of pressure varying as the power \( \frac{4}{3} \) of the incoming wave amplitude for a shock wave focusing at a smooth caustic [5]. This difference shows that the maximum overpressure evolves as a smaller power of its initial amplitude in the case of a cusp than a smooth caustic. As expected, this outlines the increased role of nonlinear effects in limiting the field amplitude at a cusp caustic, where the amplification and thus nonlinear effects are more pronounced. However, Pierce’s similitude was not entirely correct, as he also estimated that the pressure should evolve as the power \( -\frac{1}{3} \) of the geometrical parameter \( a \), while our results show that the correct power is \( -\frac{1}{6} \).

Generally, acoustic shock waves are never step shocks, but are more likely to resemble “N” waves such as observed for sonic booms, or sawtooth waves for periodic signals. Nevertheless, in most cases, the parameter \( \mu \) is small and nonlinear effects are negligible for most parts of the signal. Only the shocks will be considerably amplified and affected by nonlinearities. We can thus expect the regions of highest amplification to behave as if the shocks were step ones, so that the similitude rule applies, at least for the peak overpressures. This procedure has been applied for the prediction of sonic boom focusing at smooth caustics [15], based on the approximate numeric solution found for a step shock [13] after applying the hodograph transformation. It could be possibly generalized to cusp caustics,

![Fig. 4. Time waveform at the cusp for an incoming “N” wave — linear KZ equation.](image)
Table 1
Comparison between smooth and cusped caustics

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<th>Property</th>
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<th>Caustic cusp</th>
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</thead>
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<td>Parameter $a$ (Eq. (7))</td>
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<tr>
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<td>Transverse: $(a/k^3)^{1/4}$; longitudinal: $(a/k)^{1/2}$</td>
</tr>
<tr>
<td>Number of rays</td>
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<tr>
<td>Equation</td>
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<td>Number of independent variables</td>
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<tr>
<td>Intrinsic linear solution</td>
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<td>Singularity of the peak (linear approximation)</td>
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<tr>
<td>Similitude for a step shock</td>
<td>$\hat{p} \propto \mu^{-1/5}$</td>
<td>$\hat{p} \propto \mu^{-1/3}$</td>
</tr>
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though, in this case, no equivalent transformation is known and the step shock solution is likely to require direct numerical simulation or experimental determination.

7. Linear approximation for an incoming “N” wave

In the linear approximation ($\mu \to 0$), Eq. (16) with the boundary condition (22) can be solved analytically. After Fourier transform, the solution is sought under the form of Pearcey function, whose amplitude is determined by the boundary condition (22) using the method of stationary phase. Details are given in Appendix A. The pressure field can be finally expressed as

$$\hat{p}(\hat{t}, \hat{x}, \hat{z}) = \text{TF}^{-1} \left\{ \frac{|\omega|^{1/4}}{(2\pi)^{1/2}} \left(1 - i \text{sgn}(\omega)\right) \text{TF}(\omega) P_{\text{sgn}(\omega)}(|\omega|^{3/4} x, -|\omega|^{1/2} z) \right\},$$  \hspace{1cm} (24)

where $\text{TF}(\omega) = \int_{-\infty}^{+\infty} F(\hat{t}) e^{i\omega \hat{t}} d\hat{t}$ is the Fourier transform operator, $\text{TF}^{-1}$ the inverse Fourier transform, $P_{\pm}$ the Pearcey function and $P_{-}$ its complex conjugate.

For the particular case of an “N” wave, Eq. (24) can be expressed under a closed form at the cusp (Appendix B). There, it turns out that the signal waveform exhibits an “U” shape characteristic for sonic boom focusing (Fig. 4). The two incoming shock waves of the “N” profile are transformed by the focusing process into singular peaks, which behave as $(1 - |\hat{t}|^{-1/4}$, while the rest of the profile remains smooth and bounded. This is analogous to the smooth caustic case, but the singularity is now stronger (power $-\frac{1}{4}$ instead of power $-\frac{1}{6}$). There is also another difference: at a caustic cusp, it is clear that there is no noise before the first shock (Fig. 4). On the contrary, on a smooth caustic, the “U” waveform does not vanish completely before the first peak. This is due to the elliptic character of the Tricomi equation on the shadow zone of the caustic, while the KZ equation is hyperbolic. Such difference seems to be in qualitative agreement with the experimental measurements of sonic boom focusing made by Wanner et al. [16], for instance by comparing the time waveforms of their Fig. 7, microphone 15 or Fig. 9, microphone 13 (smooth caustic or focus) on one side, and Fig. 15, microphone 18, or Fig. 16, microphone 13 (cusp caustic or superfocus) on the other side. Naturally, the measurements display finite peak pressures, due to the limiting role played by nonlinear effects. The process is similar to the case of a smooth caustic as summarized in Table 1, but with higher nonlinear attenuation.

8. Conclusion

As proved by the similitude of Section 6 and the singularity of the linear approximation in Section 7, the focusing of a weak shock wave at a caustic cusp is essentially a nonlinear phenomena, even for small amplitude waves. The
modeling of the problem within the frame of the geometrical theory of diffraction leads to the KZ equation, with
the complete boundary condition now specified. The formulation is no longer limited to a step wave but is valid for
realistic transient waveforms such as “N” waves. The problem now appears to be numerically tractable. Similarities
and differences in the case of a smooth caustic have been outlined. However, this last case has been generalized to a
three-dimensional, inhomogeneous fluid, a challenge that remains to be taken up for the cusp caustic. The question
also remains open whether such asymptotic analysis can be extended to higher order caustics as classified by the
theory of catastrophes.

Appendix A

We are looking for the solution of the problem

$$\frac{\partial^2 \hat{p}}{\partial t \partial z} - \frac{\partial^2 \hat{p}}{\partial x^2} = 0 \tag{A.1}$$

with the boundary condition

$$\hat{p}(\hat{t}, \hat{x}, \hat{z}) \sim \frac{1}{\sqrt{\overline{t^2 + z^2} - \infty}} F(\hat{t} + \hat{a} \hat{x} + \hat{a}^2 \hat{z} - \hat{a}^4) \quad \text{in region I}, \tag{A.2}$$

where $\hat{a}(\hat{x}, \hat{z})$ is the only real root of Eq. (19).

The time Fourier transform of the pressure $\hat{p}(\hat{x}, \hat{z}, \omega)$ is a solution of the equation

$$-i\omega \frac{\partial \hat{p}}{\partial z} - \frac{\partial^2 \hat{p}}{\partial \hat{x}^2} = 0. \tag{A.3}$$

For positive pulsation $\omega > 0$, we set: $\hat{x} = \omega^{3/4} \hat{x}$ and $\hat{z} = \omega^{1/2} \hat{z}$. According to Section 3, the solution is sought to be proportional to the Pearcey function

$$\hat{p} = a(\omega) P(\hat{x}, -\hat{z}) = a(\omega) \int_{-\infty}^{+\infty} \exp(i\hat{a}^4 + i\hat{x} \hat{a} - i\hat{z} \hat{a}^2) \, d\hat{a} = a(\omega) \int_{-\infty}^{+\infty} \exp(i\hat{a}^4 - i\hat{x} \hat{a} - i\hat{z} \hat{a}^2) \, d\hat{a}. \tag{A.4}$$

A direct substitution proves that it is indeed a solution. The amplitude coefficient $a(\omega)$ is determined by the boundary
condition. Applying the method of stationary phase to Eq. (A.4) for large $\hat{x}$ and $\hat{z}$ values, the point of stationary
phase is the unique (in region I) root of $4\hat{a}^3 - 2\hat{z} \hat{a} - \hat{x} = 0$, and the pressure can be approximated by

$$\hat{p} = \frac{a(\omega) \sqrt{\pi}}{\sqrt{6\hat{a}^2 - \hat{z}}} \exp(i\hat{a}^4 - i\hat{x} \hat{a} - i\hat{z} \hat{a}^2) \exp\left(\frac{i\pi}{4}\right). \tag{A.5}$$

Comparing Eq. (A.5) with the Fourier transform of Eq. (A.2) yields the sought after the value of $a(\omega)$:

$$a(\omega) = \frac{\exp(-i\pi/4) \omega^{1/4} TF(F)(\omega)}{\sqrt{\pi}}. \tag{A.6}$$

As the pressure field is real, the negative part of the frequency spectrum is the complex conjugate of the positive one, so that the solution, Eq. (24) is finally recovered.

Appendix B

For the case of an “N” wave of duration 1:

$$F(\hat{t}) = \begin{cases} -\hat{t} & \text{if } |\hat{t}| < 1, \\ 0 & \text{else}, \end{cases} \tag{B.1}$$
Eq. (24) reduces at the cusp to

$$\tilde{p} = \frac{i}{\sqrt{2}\pi^{3/2}} \left[ \int_{-\infty}^{0} (1 + i)(X - iY)|\omega|^{1/4} \frac{d}{d\omega} \left( \frac{\sin \omega}{\omega} \right) \exp(-i\omega\tilde{t}) d\omega \right] + \left[ \int_{0}^{+\infty} (1 - i)(X + iY)|\omega|^{1/4} \frac{d}{d\omega} \left( \frac{\sin \omega}{\omega} \right) \exp(-i\omega\tilde{t}) d\omega \right],$$

(B.2)

with \( P(0, 0) = X + iY = \frac{1}{2}[\Gamma(\frac{1}{2})(\cos(\pi/8) + i \sin(\pi/8))] \).

After integration by parts, Eq. (B.2) reduces to

$$\tilde{p} = \frac{\sqrt{2}}{\pi^{3/2}} \left[ \int_{0}^{+\infty} \frac{\sin \omega}{\omega^{3/4}} \left( (Y - X) \left( \frac{\cos(\omega\tilde{t})}{4\omega} - i \sin(\omega\tilde{t}) \right) - (X + Y) \left( \frac{\sin(\omega\tilde{t})}{4\omega} + i \cos(\omega\tilde{t}) \right) \right) d\omega, \right.$$ (B.3)

or

$$\tilde{p} = \frac{\sqrt{2}}{\pi^{3/2}} \left[ \int_{0}^{+\infty} \frac{\sin \omega}{\omega^{3/4}} \left( (Y - X) \left( \frac{\cos(\omega\tilde{t})}{4\omega} - i \sin(\omega\tilde{t}) \right) - (X + Y) \left( \frac{\sin(\omega\tilde{t})}{4\omega} + i \cos(\omega\tilde{t}) \right) \right) d\omega \right].$$

(B.4)

Using the formula [32] for \( a > 0, b > 0, -1 < \Re \mu < 1 \):

$$S(\mu, a, b) = \int_{0}^{+\infty} x^{\mu - 1} \sin(ax) \sin(bx) dx = \frac{1}{2} \cos \left( \frac{\mu \pi}{2} \right) \Gamma(\mu)(|a - b|^{-\mu} - (a + b)^{-\mu}),$$

$$C(\mu, a, b) = \int_{0}^{+\infty} x^{\mu - 1} \sin(ax) \cos(bx) dx = \frac{1}{2} \sin \left( \frac{\mu \pi}{2} \right) \Gamma(\mu)(|a + b|^{-\mu} + \text{sgn}(a - b)|a - b|^{-\mu}),$$

(B.5)

yields the final result

$$\tilde{p} = \frac{\sqrt{2}}{\pi^{3/2}} \left[ \frac{Y - X}{4} C \left( -\frac{3}{4}, 1, |\tilde{t}| \right) - \text{sgn}(\tilde{r}) \frac{X + Y}{4} S \left( -\frac{3}{4}, 1, |\tilde{t}| \right) \right] + \left[ \tilde{r} (X - Y) S \left( \frac{1}{4}, 1, |\tilde{t}| \right) - \tilde{r} (X + Y) C \left( \frac{1}{4}, 1, |\tilde{t}| \right) \right].$$

(B.6)

From Eq. (B.5), it is clear that the waveform exhibits a singularity of the type \((1 - |\tilde{t}|)^{-1/4}\) associated with the focusing of the two weak shock waves of the incoming “N” wave.

References