Sonic boom in the shadow zone: A geometrical theory of diffraction

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Geometrical acoustics predicts the amplitude of sonic booms only within the carpet. Inside the geometrical shadow zone, a nonlinear, geometrical theory of diffraction in the time domain is proposed. An estimation of magnitude orders shows that nonlinear effects are expected to be small for usual sonic booms. In the linear case, the matching to geometrical acoustics yields an analytical expression for the pressure near the cutoff. In the shadow zone, it can be written as a series of creeping waves. Numerical simulations show that the amplitude decay of the signal compares favorably with Concorde measurements, while the magnitude order of the rise time is correct. The ground impedance is shown to influence the rise time and peak amplitude of the signal mostly close to the cutoff. In the case of a weakly refractive atmosphere (low temperature gradient or downwind propagation), the transition zone about the cutoff is large, the transition is smooth, and the influence of ground absorption is increased. © 2002 Acoustical Society of America.

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I. INTRODUCTION

Sonic boom is the noise associated with an aircraft flying at supersonic speeds. According to the classical theory (Whitham, 1952, 1956; Guiraud, 1965; Hayes et al., 1969), sonic boom waveforms can be predicted within the frame of geometrical acoustics; sound propagates in the atmosphere along classical acoustic rays. These rays are launched perpendicular to the Mach cone formed by the airplane, but then deviate by refraction. In a quiescent, vertically stratified atmosphere, refraction leads to the formation of a shadow zone; only a limited area of the ground—the so-called “carpet”—is touched by direct rays (Fig. 1). Beyond the carpet edge, geometrical acoustics predicts no signal. Several computer codes [Hayes et al., 1969; Thomas, 1972; Bass et al., 1987 (SHOCKN); Robinson, 1991 (ZEPHYRUS); Cleveland, 1995 (THOR); Plotkin, 1998 (PCBoom3)] have been developed for predicting the sonic boom. They generally give good agreement with measured data, except close to the carpet edge where pressure amplitude is systematically overpredicted (Maglieri and Plotkin, 1995; Plotkin, 1998; Downing, 1998). Moreover, no computer code predicts sonic booms inside the shadow zone.

The exact solution in a particular case (Berry and Daigle, 1988; Pierce, 1989)—a point source in a stratified atmosphere with a specified sound speed profile—elucidates the physics of diffraction near the carpet edge: the limiting ray forming the shadow boundary sheds off creeping rays at the point of tangency with the ground (Fig. 2). Inside the shadow zone, these creeping waves propagate along the ground surface. Above the ground, the sound field emanates from rays diffracted by these creeping waves. The signal attenuation is explained by the energy lost as the creeping waves shed off more and more diffracted rays while propagating further into the shadow zone.

The solution of Pierce (1989) can be extended to an impulse signal (Raspet and Franke, 1988) by means of Fourier transform. However, Pierce’s solution is strictly valid only for linear sound propagation from a point source, and cannot be extended without precaution to the sonic boom case. The first reason is that a supersonic aircraft radiates not spherical but conical wavefronts. More important, nonlinear effects severely affect sonic boom propagation. In order to describe the sonic boom in the shadow zone, it is necessary to separate two distinct steps: (1) nonlinear propagation in the atmosphere (from the source to the carpet cutoff) described by ray acoustics, and (2) diffraction near the cutoff. For this second step, we need a proper way for matching locally the ray description over the carpet to the creeping waves series in the shadow zone. For this article we proceed in a way similar to Bouche and Molinet (1994), who studied the shadow zone of a convex body in a homogeneous medium. We here extend their method to the case of an upward refracting atmosphere, and generalize it to a transient signal. It is also necessary to prove that nonlinear effects are negligible in the shadow zone, contrary to propagation above the carpet.

In Sec. II, a model equation—the unsteady Tricomi equation (Coulouvrat, 1997)—is derived for the sound field inside the shadow zone according to the geometrical diffraction process previously outlined. This equation includes nonlinearities. An estimation of the magnitude order of nonlinear propagation in the geometrical acoustics approximation inside the carpet (Sec. IV). This yields an analytical expression for the pressure field about the carpet edge, as a function of the incident field and ground impedance. This expression takes the form of a Fourier transform of the Fock integral in the frequency domain. Inside the shadow zone, this equation can be extended to an impulse signal (Raspet and Franke, 1988) by means of Fourier transform. However, Pierce’s solution is strictly valid only for linear sound propagation from a point source, and cannot be extended without precaution to the sonic boom case. The first reason is that a supersonic aircraft radiates not spherical but conical wavefronts. More important, nonlinear effects severely affect sonic boom propagation. In order to describe the sonic boom in the shadow zone, it is necessary to separate two distinct steps: (1) nonlinear propagation in the atmosphere (from the source to the carpet cutoff) described by ray acoustics, and (2) diffraction near the cutoff. For this second step, we need a proper way for matching locally the ray description over the carpet to the creeping waves series in the shadow zone. For this article we proceed in a way similar to Bouche and Molinet (1994), who studied the shadow zone of a convex body in a homogeneous medium. We here extend their method to the case of an upward refracting atmosphere, and generalize it to a transient signal. It is also necessary to prove that nonlinear effects are negligible in the shadow zone, contrary to propagation above the carpet.

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In Sec. II, a model equation—the unsteady Tricomi equation (Coulouvrat, 1997)—is derived for the sound field inside the shadow zone according to the geometrical diffraction process previously outlined. This equation includes nonlinearities. An estimation of the magnitude order of nonlinear versus diffraction effects shows that in usual cases, nonlinearities are expected to be small (Sec. III). In the linear case, it is then possible to properly match the solution of the unsteady Tricomi equation to the geometrical acoustics approximation inside the carpet (Sec. IV). This yields an analytical expression for the pressure field about the carpet edge, as a function of the incident field and ground impedance. This expression takes the form of a Fourier transform of the Fock integral in the frequency domain. Inside the shadow zone, this equation can be extended to an impulse signal (Raspet and Franke, 1988) by means of Fourier transform. However, Pierce’s solution is strictly valid only for linear sound propagation from a point source, and cannot be extended without precaution to the sonic boom case. The first reason is that a supersonic aircraft radiates not spherical but conical wavefronts. More important, nonlinear effects severely affect sonic boom propagation. In order to describe the sonic boom in the shadow zone, it is necessary to separate two distinct steps: (1) nonlinear propagation in the atmosphere (from the source to the carpet cutoff) described by ray acoustics, and (2) diffraction near the cutoff. For this second step, we need a proper way for matching locally the ray description over the carpet to the creeping waves series in the shadow zone. For this article we proceed in a way similar to Bouche and Molinet (1994), who studied the shadow zone of a convex body in a homogeneous medium. We here extend their method to the case of an upward refracting atmosphere, and generalize it to a transient signal. It is also necessary to prove that nonlinear effects are negligible in the shadow zone, contrary to propagation above the carpet.
zone, the Fock integral can be written as an infinite series of creeping waves. Then, numerical simulations are performed for an incident \( "N" \) wave, over several different ground materials (Sec. V). The model of Attenborough (1983) is chosen for describing the ground impedance, as this model is considered a good description of rigid-frame porous materials for which data are available in the literature for typical outdoor ground surfaces. The numerical simulations are finally compared to Concorde measurements (Parmentier et al., 1973) (Sec. VI). Results are established only for a quiescent, vertically stratified atmosphere. However, the influence of wind stratification will be discussed qualitatively.

Despite the extensive literature about sonic boom, little attention has been paid to diffraction effects near the carpet edge and inside the shadow zone. To our knowledge, the most recent theoretical work about sonic boom in the shadow zone was published by Onyeowu in 1975. Although relying also on the geometrical theory of diffraction briefly outlined above, Onyeowu’s model suffers from several drawbacks which are overcome in the present work. First, Onyeowu did not evaluate the importance of nonlinear effects in the shadow zone. Second, he took into account only the first mode of the creeping wave series, an approximation which was not quantified. More important, the matching with geometrical acoustics was performed by Onyeowu through an empirical reflection coefficient varying between 1 (at the cutoff) and 2 (geometrical acoustics). To be consistent with the classical expression of the reflection coefficient of a grazing plane wave, the arbitrary value of 1 at the cutoff requires that the ground impedance is equal to zero. Therefore, his creeping waves dispersion relation is that over a perfectly

\[ \psi = \psi_0 + \psi_1 + \psi_2, \]

\[ \psi_0 = \frac{\partial P}{\partial x} \frac{\partial P}{\partial \tau} \]

\[ \psi_1 = \frac{1}{c_0} \frac{\partial}{\partial \tau} \left( \frac{\partial P}{\partial x} \right) \]

\[ \psi_2 = \frac{1}{c_0} \frac{\partial}{\partial \tau} \left( \frac{\partial P}{\partial x} \right) \]

压力释放表面，一个假设这与在地面计算的压力场的假设不一致。结果，他不能考虑到地面的有限地阻尼。最终，没有比较是与飞行数据的计算结果。影响的扩散效应在信号的上升时间的影响也是未被探讨，这仍然是一个关键的参数以避免对户外声波的干扰。

II. A NONLINEAR, GEOMETRICAL THEORY OF DIFFRACTION IN THE SHADOW ZONE

To estimate nonlinear effects inside the shadow zone, we propose a new, nonlinear, time-domain formulation of diffraction effects in the shadow zone. The detailed derivation of the generalized ‘‘unsteady’’ Tricomi equation has been published elsewhere (Coulouvrat, 1997) but will be briefly recalled here, as it is the basis for estimating nonlinear effects, and then deriving the analytical expression of the pressure field. According to the geometrical theory of diffraction, the acoustical path from the source \( S \) to the current observation point \( M \) in the shadow zone is the path \( SO \) from the source to the cutoff along the limiting ray, plus the ground path \( OC \) from the cutoff to the contact point of the diffracted ray with the ground, plus the path \( CM \) from the contact point to the current observation point along the diffracted ray (Fig. 2). Path \( SO \) is given by classical geometrical acoustics in a heterogeneous atmosphere. Provided altitude \( z \) is small compared to the radius of curvature of the limiting ray \( R \), the propagation time \( \tau \) from the carpet edge \( O \) to the current point \( M \) is

\[ \tau = \frac{x^*}{c_0} + \frac{\partial P}{\partial x} \frac{\partial P}{\partial \tau} \]

\[ c_0 \] being the sound speed at the ground. The radius \( R \) is related to the sound speed vertical profile \( c(z) \) by

\[ R = -c_0 \left( \frac{\partial c}{\partial z} \right) \]

The present theory is limited to the case of an upward-refracting theory for which the sound speed gradient is negative at the altitude of the ground. The distance \( x^* \) between the cutoff and the projection \( P \) of the observation point on the ground is measured in the direction of the limiting ray. For steady horizontal flights, this direction makes a constant angle \( \phi = \sin^{-1} \left( c_0 / \sqrt{M} \right) \) with the perpendicular to the carpet edge (Fig. 3), \( M \) being the Mach number and \( c_{av} \) the sound speed at the flight altitude.

Following a procedure similar to the one used for describing the vicinity of caustics (Pierce, 1989), Eq. (1) leads us to introduce the following dimensionless variables, which are better suited to the physics of the problem:

\[ \tilde{\tau} = \left( \frac{t - x^*}{c_0} \right) / \tau_{inc} \]

\[ \tilde{z} = \left( \frac{2c_0^2 \tau_{inc} R^2}{\tau_{inc}} \right) / \tilde{\tau} \]

where \( \tau_{inc} \) and \( P_{inc} \) are, respectively, the duration and the maximum amplitude of the incident pressure field at the cutoff (the field given by geometrical acoustics at the cutoff as if there was no ground).

Variable \( \tilde{\tau} \) is introduced for the propagation time along the ground, \( \tilde{z} \) for the diffraction boundary-layer thickness above the ground, and \( \tilde{x} \) for the field attenuation as it penetrates into the shadow zone. Rewriting the fluid mechanics

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**FIG. 1.** The sonic boom carpet.

**FIG. 2.** Geometry of the problem in the shadow zone.
equations with this new system of coordinates, the dimensionless pressure field \( \tilde{p}_a \) can be shown to satisfy Eq. (3) in the shadow zone:

\[
\frac{\partial^2 \tilde{p}_a}{\partial t^2 - \nabla^2} + \frac{\mu}{\partial t} \frac{\partial \tilde{p}_a}{\partial z^2} = 0,
\]

with \( \rho_0 \) the air density at ground level and \( \beta = 1 + B/2A \) the nonlinearity parameter (\( \beta = 1.2 \) for air). The dimensionless parameter \( \mu \) defined by

\[
\mu = \frac{B}{\rho_0 c_0} \left( \frac{R}{T_{inc} c_0} \right)^{2/3},
\]

is a measurement of quadratic nonlinear effects relative to diffraction effects. In Eq. (3), cubic nonlinear terms and second-order diffraction effects have been neglected. The right-hand side of Eq. (3) is identical to the nonlinear Tricomi equation obtained by Guiraud (1965) for describing the nonlinear sound field in the vicinity of caustics, while the left-hand side describes the field evolution as it penetrates into the shadow zone. The generalized nonlinear Tricomi equation (3) can easily be shown to be similar to the unsteady, small perturbation transonic potential equation in a stratified atmosphere. Therefore it may be called an “unsteady” Tricomi equation; here the “unsteady” term \( \frac{\partial^2 \tilde{p}_a}{\partial t \partial z} \) describes the wave penetration inside the shadow zone.

Boundary conditions are of course different from that of aerodynamics, and are the following: (i) the pressure field vanishes at infinite times:

\[
\tilde{p}_a(t = \pm \infty, x, z) = 0,
\]

(ii) it satisfies the impedance condition at the ground:

\[
\tilde{p}_a = i \omega T_{inc} \left( \frac{2c_0 T_{inc}}{R} \right)^{1/3} \frac{Z_0}{Z_0} \frac{\partial \tilde{p}_a}{\partial z} \text{ at } z = 0
\]

in Eq. (5b), \( \hat{\tilde{p}}_a \) is the time Fourier transform of the pressure field, \( Z_0 \) and \( Z_s \) are respectively the air and ground surface impedances, and \( \omega/2\pi \) the frequency, and (iii) it satisfies a radiation condition far from the ground in the shadow zone,

\[
\tilde{p}_a \sim \tilde{y}^{-1/4} G(\tilde{x}, \tilde{z} - 2\tilde{z}^{3/2}/3) + \tilde{y}^{-1/4} \frac{\partial \tilde{p}_a}{\partial t} (\tilde{x}, \tilde{z} \to + \infty) = 0.
\]

The matching condition to geometrical acoustics will be detailed in Sec. IV.

### III. MAGNITUDE ORDERS

The parameter \( \mu \) [Eq. (4)] measures the relative magnitude order of nonlinear effects compared to diffraction around the cutoff and inside the shadow zone. Nonlinear effects are dominant during geometrical propagation from the aircraft down to the ground, and surely are negligible deep inside the shadow zone where amplitude is small and diffraction-induced attenuation dominant. The question remains on the importance of nonlinear effects around the cutoff. A moderate value of parameter \( \mu \) would mean that nonlinearities are intimately coupled to diffraction there. The value of parameter \( \mu \) is evaluated in Table I in two cases: first for a standard atmosphere, second for a low temperature-gradient atmosphere. As an example, we chose the case of an atmosphere having a 5 °C ground temperature, a constant gradient of \(-0.1 \) °C/km over 1 km above the ground, and then a constant gradient of \(-6.14 \) °C/km up to 11 km (Fig. 4). Above, it is identical to the standard atmosphere. Such an atmosphere can be viewed as a simple model for a quiet winter morning. According to Eq. (4), a small similar to the one satisfied by the sound field reflected over a caustic (Guiraud, 1965; Seebass, 1971):

\[
\tilde{p}_a \sim \tilde{y}^{-1/4} G(\tilde{x}, \tilde{z} - 2\tilde{z}^{3/2}/3),
\]

where the function \( G(\tilde{x},.) \) is the (undetermined) outgoing waveform emanating from diffracted rays far over the ground at distance \( \tilde{x} \). Equation (5c) can be equivalently written as a radiation condition specifying that there is no incident wave inside the shadow zone:

\[
\tilde{p}_a \sim \tilde{y}^{-1/4} \frac{\partial \tilde{p}_a}{\partial t} (\tilde{x}, \tilde{z} \to + \infty) = 0.
\]

### TABLE I. Parameter estimation (aircraft flying at 11,000 m and Mach 2.0).

<table>
<thead>
<tr>
<th>Atmosphere</th>
<th>Standard</th>
<th>Low gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carpet width (km)</td>
<td>79</td>
<td>239</td>
</tr>
<tr>
<td>Incident pressure amplitude (Pa)</td>
<td>50</td>
<td>10</td>
</tr>
<tr>
<td>Parameter ( \mu )</td>
<td>0.03</td>
<td>0.1</td>
</tr>
<tr>
<td>Boundary-layer thickness ( H ) (m)</td>
<td>200</td>
<td>790</td>
</tr>
<tr>
<td>Matching zone width ( W ) (km)</td>
<td>4.8</td>
<td>44</td>
</tr>
</tbody>
</table>

FIG. 3. The sonic boom carpet (view from top).

FIG. 4. Standard and low gradient atmosphere.
temperature gradient, and consequently a large radius of curvature $R$, is expected to increase the relative importance of nonlinear effects in the shadow zone. However, a small temperature gradient also leads to an increase of the carpet width and a decrease of the incident pressure amplitude at the cutoff by geometrical attenuation, thus counterbalancing the decrease of the rays curvature. In order to take this into account, we computed the ray-tube area at the cutoff for the two different atmospheres; the ray-tube area for the low temperature gradient was found to be about 25 times larger. According to linear geometrical acoustics, we choose an incident pressure amplitude at the cutoff five times smaller. Additional decay due to nonlinear attenuation over a longer propagation path is expected to dampen even further the field at the cutoff. Compared values for the two different cases are collected in Table I. Results show that, in both cases, nonlinear effects are expected to be relatively small. However, this conclusion can be viewed only as provisional. The two cases studied in this article cannot be viewed as really representative of all atmospheric situations. Especially, the influence of wind gradients should be taken into account as indicated below. There could be some (probably rare) atmospheres almost uniform very near the ground but with very sharp gradients (of either temperature or wind) in altitude, which could lead to larger values. A statistical analysis relying on a long-term meteorological database is planned to be completed in a way similar to previous studies on sonic boom carpet widths (Heimann, 2001). Such a study would determine the probability (if any) of occurrence of meteorological situations implying significant nonlinear effects near the cutoff.

In Table I, two other characteristic lengths also are evaluated: the thickness $H$ of the diffraction boundary layer above the ground, and the width $W$ of the matching zone on each side of the cutoff. The thickness $H$ is defined as the height at which the arrival time of the diffracted signal at point $M$ inside the shadow zone differs from more than $T_{\text{inc}}$ from the arrival time at point $P$ on the ground (Fig. 2). The transition zone between the carpet, where geometrical acoustics is valid, and the true shadow zone in which the sound field decays exponentially, can be estimated by $W$ (Fig. 3). This gives

$$H = \left( \frac{9Rc_0^2T_{\text{inc}}^2}{8} \right)^{1/3},$$

$$W = \cos \psi(c_0T_{\text{inc}}R^2)^{1/3}. \quad (6)$$

It is to be noted that the characteristic width $W$ is also Mach dependent, and increases with the Mach number. In Table I we note the large increase of the width of the matching zone in the low-gradient atmosphere. A large radius of curvature (weakly refractive atmosphere) leads to a smooth transition near the cutoff and a slow decay in the shadow zone, while a small radius of curvature (strongly refractive atmosphere) leads to a sharp transition and a rapid decay in the shadow zone. Such a qualitative conclusion can explain the asymmetry of the lateral peak amplitude distribution measured by Maglieri and Plotkin (1995) if we assume a windy atmosphere. An upwind propagation (small radius of curvature) leads to a shorter carpet, a sharper transition, and a stronger decay in the shadow zone, while a downwind propagation (large radius of curvature) leads to a wider carpet, a smoother transition, and a slower decay in the shadow zone. These qualitative effects are illustrated on Fig. 5. They are observed in the measurements reported by Maglieri and Plotkin (their Figure 16) for an XB-70 flying at Mach 2.0, if we assume an upwind propagation on the left side of the figure and a downwind propagation on the right side.

The detailed diffraction process in the shadow zone of a stratified, windy atmosphere remains rather intricate, though an approximate generalization of Pierce’s solution was given recently by Li et al. (1998). One difficulty is that, if the ground wind is not constant, the creeping waves propagate over the ground along curved rays deviated by lateral winds. Moreover, in the windy case, rays are not normal anymore to the wavefront, so that Eq. (1) is not strictly valid anymore. However, if the atmosphere is vertically stratified, creeping waves propagate along straight rays, and the diffracted rays are normal to the wavefront at the ground level. In this case, we expect the diffraction process to be similar to the quiescent case, but with a modified radius of curvature taking into account the wind stratification:

$$R = -\frac{c_0 + \tilde{u}(0) \cdot \tilde{n}}{[(dc/dz)(0) + (du/dz)(0) \cdot \tilde{n}]} + \frac{1}{2}, \quad (7)$$

where $\tilde{u}(z)$ is the horizontal wind speed and $\tilde{n}$ is the unit horizontal vector along the $Ox$ axis (oriented towards the shadow zone). As wind gradients are frequently comparable in magnitude order to temperature gradients, significant wind effects can thus be simply incorporated into the present model. Especially, very large radii of curvature are possible for downwind propagation, leading to effects similar to those observed for small temperature gradients [however, if the wind gradient is too large, the radius (7) will be negative in the case of downwind propagation, and the geometrical shadow zone will not exist any more].
IV. MATCHING TO GEOMETRICAL ACOUSTICS

In the linear case, it is possible to find an analytical expression of the sound field in the shadow zone by a proper matching to the geometrical acoustics approximation over the carpet. We proceed in a way similar to Bouche and Molinet (1994), who study the shadow zone of a convex body in a homogeneous medium. Their method is applied here to the case of an upward-refracting atmosphere. They suggest splitting the sound field into an incident geometrical part $\tilde{P}_{inc}^{\text{geo}}$ (the field given by geometrical acoustics at the cutoff as if there was no ground), and a diffracted part $\tilde{P}_{\text{dif}}$ which takes into account ground effects and diffraction. First, an analytical approximation of the incident field near the cutoff is derived. The eikonal function can be evaluated along the limiting ray by a straightforward Taylor expansion:

$$
\Psi(\bar{l}) = \frac{1}{c_0} (l + l^3/6R^2 + O(l^4/R^3)),
$$

where $l$ is the curvilinear coordinate along the limiting ray. At the same order of approximation, the equation of the wavefront cutting the limiting ray at the point of curvilinear coordinate $l$ is

$$
l = x + \frac{x z}{R} - \frac{x^3}{3R^2} + O(x^4/R^3),
$$

so that the eikonal function in the vicinity of the carpet edge can be approximated by

$$
\Psi(\bar{x}) = \frac{1}{c_0} \left( x + \frac{x z}{R} - \frac{x^3}{6R^2} + O(x^4/R^3) \right).
$$

Consequently, the incident pressure field is given near the cutoff by

$$
\tilde{P}_{inc} = F(\bar{l} - \bar{x} x + \bar{x}^3/3),
$$

$F(\bar{l})$ being the (dimensionless) time waveform at the cutoff. It is easy to check that expression (11) is an exact solution of the linear Tricomi equation. It is then possible to determine by Fourier transforms the analytical expression of the acoustic field as a Fock integral (Pierce, 1989) in the vicinity of the carpet edge. We set

$$
\tilde{P}_{inc} = \frac{1}{c_0} \int_{-\infty}^{+\infty} \hat{P}(\tilde{\omega}, \tilde{K}, \tilde{z}) \exp(ik_0(\tilde{x} - \tilde{\omega} \tilde{r})) d\tilde{K},
$$

with $TF^{-1}(F)(\tilde{\omega})$ the inverse Fourier transform of $F$, $A_i$ the Airy function and $A_i'$ its derivative. The diffracted component is proportional to

$$
A_i\left( \frac{\tilde{K} \text{sgn}(\tilde{\omega})}{|\tilde{\omega}|^{1/3}} - |\tilde{\omega}|^{2/3} \right) \exp\left( \frac{2i \text{sgn}(\tilde{\omega}) \tau}{3} \right),
$$

which is the solution of the (linear) unsteady Tricomi equation that satisfies the radiation condition Eq. (5d). The proportionality constant is determined by the impedance boundary condition Eq. (5b) satisfied by the total field. This completely determines the pressure field close to the cutoff. Returning to physical variables, one has (Coulouvrat, 1998)

$$
p_a(x^*, z, t) = TF\left( \frac{2c_0 R^2}{\omega} \right)^{1/3} \hat{P}(\omega, x^*, z) TF^{-1}(F)(\omega),
$$

with

$$
\hat{P} = \int_{-\infty}^{+\infty} dk \left( A_i(\tau - \frac{z}{l(\omega)}) - \frac{e^{2i\pi/3} A_i'(\tau) - e^{q} A_i(\tau e^{2i\pi/3})}{A_i'(b_n)^2 - b_n A_i(b_n)^2} \right) \exp(ikx^*),
$$

where $l(\omega) = (c_0 R/2\omega^2)^{1/3}$ is a characteristic boundary-layer thickness at frequency $\omega$. $\tau(\omega, k) = 2(\omega/c_0)(k - \omega/c_0)^2$, $e = \text{sgn}(\omega)$, and $q(\omega) = (Z_0/Z_\infty(\omega))(\omega R/2c_0)^{1/3}$ is a measurement of finite ground-impedance effects at frequency $\omega$.

In the shadow zone, we deduce from the residue theorem the expansion of integral (15) into a series of creeping waves:

$$
p_a = TF\left( \frac{2c_0 R^2}{\omega} \right)^{1/3} \hat{P}(\omega, x^*, z) TF^{-1}(F)(\omega)
$$

$$
\times \sum_n \exp(ik_n x^*) \frac{A_i(b_n - \tau/\omega) e^{2i\pi/3}}{A_i'(b_n)^2 - b_n A_i(b_n)^2}.
$$

The residues $(b_n(\omega))_{n \in N}$ are the complex roots of the equation

$$
A_i'(b_n) + e^{i\pi/3} q(\omega) A_i(b_n) = 0,
$$

related to the complex wave number $(k_n(\omega))_{n \in N}$ of the creeping waves by

$$
k_n = \frac{\omega}{c_0} \left( 1 + \frac{1}{2} \exp\left( \frac{-2i\pi}{3} \right) \frac{b_n}{(\omega\omega/c_0)^{\pi}} \right).
$$

It is remarkable that, in the general case, the coefficients of the creeping waves series [Eq. (16)] are identical as those found by Berry and Daigle (1988) for a line source whose height tends to infinity. This means that, for a source located outside the diffraction boundary-layer, the way sound diffracts at the cutoff is “universal,” in the sense that it depends only on atmosphere properties near the ground and ground impedance, but not on the source. The dimensionless formulation even shows that the case of an incident “N” wave on a rigid ground can be made independent of any parameters. This property has been used by Truphème and Coulouvrat (1999) for comparisons with “BoomFile” experiments made by USAF (Lee and Downing, 1991).

V. SOUND FIELD AT THE CUTOFF

At the cutoff and on a perfectly rigid ground, the series (16) simplifies to
The incident pressure field on a rigid ground is thus amplified by a factor 1.399, instead of a factor 2 (mirror reflection) deep inside the carpet. This classical result (Logan and Yee, 1962) proves that (i) the value of 1 postulated by Onyeowu for his empirical reflection coefficient is not supported by theory, (ii) if only the first term is taken into account in the series expansion (19), one gets a value equal to 1.832 instead of 1.4, thus overestimating the cutoff amplitude by 31%.

However, the assumption of a rigid surface is not very realistic, especially at medium and high frequencies, for which parameter $q(\omega)$ cannot be neglected any more. Finite impedance effects lead to absorption by the porous ground material as sound grazes over the geometrical carpet. This results in lower amplitudes and finite rise times at the cutoff, and a larger attenuation in the shadow zone. For describing the ground material properties, the model of Attenborough (1983) gives the ground impedance as a complex, frequency-dependent function:

$$Z_s(\omega) = \frac{aZ_0}{\Omega \sqrt{1 + (\gamma - 1) T(\sqrt{iPr^{1/2}}\lambda_p)[1 - T(\sqrt{i\lambda_p})]}}$$

where $a$ is the tortuosity ($a>1$), $\Omega$ is the porosity ($\Omega<1$), $\gamma=1.4$ is the ratio of the specific heats, and $Pr=0.724$ is the Prandtl number. The parameter $\lambda_p = \sqrt{8 \alpha_0 \sigma_f \sigma_f \sigma_f}$ is the ratio of the mean pore size to the acoustic boundary layer thickness inside a pore, $\sigma_f$ being the static flow resistivity and $s_f$ a dynamical shape parameter. Function $T(x)$ is equal to $T(x) = 2J_1(x)/xJ_0(x)$ with $J_n(x)$ the $n$th Bessel functions. Among the four parameters characterizing the material $(a, \Omega, \sigma_f, s_f)$, only the shape parameter is not directly measurable, but it varies little between 0.5 and 1. At low frequencies, acoustical absorption in the porous material can be described by a single parameter, the effective resistivity $\sigma_{eff} = s_f^3 \alpha / \Omega$. For numerical simulations, we chose four outdoor ground surfaces, whose parameter values found in the literature (Attenborough, 1983, 1985) are recalled in Table II. Ground effects were evaluated for an incident “N” wave of duration 0.27 s and of normalized unit amplitude 1 over the four ground surfaces selected previously. Results are collected in Table III for the two different atmospheres described in Sec. III. Ground impedance effects are all the more pronounced as the (effective) flow resistivity of the ground material is smaller, as can be seen in Table III. However, even for a relatively rigid soil such as a barren sandy plain and a standard atmosphere, the signal rise time produced by ground porosity at the cutoff is of order 10 ms. It is comparable or longer than typical rise times resulting from propagation into a turbulent atmosphere over the carpet (generally between 1 and 10 ms, Bass et al., 1998). Consequently, one can conclude that ground impedance effects must be taken into account to predict the rise time of sonic booms close to the cutoff on each side of the carpet edge. A low temperature gradient dramatically increases the influence of ground absorption, because of the increased width of the transition zone through which sound grazes over the ground. Very large rise times are forecast, which means much less annoying booms for outdoor hearing (annoyance for sonic boom indoors is primarily due to building rattle which depends on duration and peak pressure but very little on rise time). In a windy atmosphere, a similar effect can be expected on the downwind carpet side (long and smooth signals are reported by Maglieri and Plotkin (1995) on what we believe to be the downwind side of the carpet). On the upwind side, influence of ground absorption will be smaller, leading to a cutoff field closer to the rigid ground case. One important conclusion is that ground diffraction and absorption effects are also important inside the carpet, over the distance $W$ that can vary between a few kilometers to several tens of kilometers, depending on the atmospheric conditions. It is well known that the geometrical theory of diffraction used in the present model is not always valid deep inside the shadow zone, because of sound scattering by turbulence (Daigle et al., 1986). However, the present model shows that diffraction effects can lead to large effects (long amplitudes and long rise times) also in regions where the theory is well established.

### TABLE II. Parameters for outdoor ground surfaces.

<table>
<thead>
<tr>
<th>Ground type</th>
<th>$\Omega$</th>
<th>$a$</th>
<th>$s_f$</th>
<th>$\sigma_0$ (kPa s/m$^2$)</th>
<th>$\sigma_{eff}$ (kPa s/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red-pine forest floor</td>
<td>0.529</td>
<td>1.65</td>
<td>0.9</td>
<td>50</td>
<td>77</td>
</tr>
<tr>
<td>Grass covered field</td>
<td>0.4</td>
<td>1.58</td>
<td>0.75</td>
<td>300</td>
<td>422</td>
</tr>
<tr>
<td>Bare sandy terrain</td>
<td>0.269</td>
<td>1.39</td>
<td>0.725</td>
<td>366</td>
<td>715</td>
</tr>
<tr>
<td>Barren sandy plain</td>
<td>0.379</td>
<td>1.27</td>
<td>0.725</td>
<td>1820</td>
<td>2524</td>
</tr>
</tbody>
</table>

### TABLE III. Calculated amplification factor and rise time at the cutoff; standard atmosphere/low gradient atmosphere.

<table>
<thead>
<tr>
<th>Ground nature</th>
<th>Amplification factor</th>
<th>Rise time (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid</td>
<td>1.40 (1.40)</td>
<td>0(0)</td>
</tr>
<tr>
<td>Barren sandy plain</td>
<td>1.24 (1.04)</td>
<td>9.2 (25.7)</td>
</tr>
<tr>
<td>Bare sandy terrain</td>
<td>1.17 (0.88)</td>
<td>14.8 (39.9)</td>
</tr>
<tr>
<td>Grass-covered field</td>
<td>1.13 (0.81)</td>
<td>18.1 (46.8)</td>
</tr>
<tr>
<td>Red-pine forest floor</td>
<td>0.96 (0.54)</td>
<td>33.0 (73.5)</td>
</tr>
</tbody>
</table>

FIG. 6. Concorde test flights.
VI. COMPARISONS WITH CONCORDE TEST FLIGHTS

The sonic boom inside the shadow zone as predicted by theory will now be simulated numerically and compared to Concorde sonic boom measurements. Computation of sonic booms over a finite impedance ground surface requires several successive steps. (1) Calculate the Fourier transform of the incident pressure field by means of fast discrete Fourier transform algorithms (FFT). (2) For each frequency, solve the dispersion equation (17) for the selected number of creeping waves. (3) Calculate the series coefficient appearing in Eq. (16) and perform the series summation. (4) Compute the time signal by an inverse FFT.

As typical incident booms resembles “N” waves with two steep shock waves, it is necessary to retain a large number of frequencies (∼2^{13}). As the creeping waves series is slowly convergent, a high number of modes is necessary (∼50). Consequently, step (2) in the numerical procedure is numerically time consuming if the ground surface is not perfectly rigid. For each frequency and each creeping mode, the dispersion equation is solved according to the procedure described by Raspet et al. (1991), by a Newton iterative algorithm. The frequency is increased step-by-step. The initial point of the Newton algorithm is the solution computed for the previous frequency and same mode number. For the lowest frequency, the initial point is the rigid case solution.

In 1973, Concorde test flights were conducted near the city of Biscarrosse (Gironde, France) along the Atlantic Coast. Concorde flew over the Atlantic Ocean in a north/south direction, and sonic boom signals were registered by microphones located in the Landes region, along two south/north and east/west axes (Fig. 6) (Parmentier et al., 1973). Test flights were performed at different distances from the coast and at different Mach numbers. For some of the flights, Concorde trajectory was such that the microphones were partly or totally in the estimated shadow zone.

Numerical results are presented in Figs. 7–12 and compared to some Concorde sonic boom measurements. Figure 7 shows the computed maximum overpressures and rise times versus distance to the cutoff over a rigid ground. Results are compared to two Concorde test flights, one (indicated by circles) for which a few microphones were located inside the shadow zone close to the carpet edge, and one (indicated by stars) for which all microphones were located relatively deep inside the shadow zone. Numerical results were obtained for a standard atmosphere and an incident “N” wave of duration 0.27 s and 21 Pa amplitude. Simulations show that
(i) the maximum overpressure decays almost exponentially with distance in the shadow zone; and
(ii) the rise time increases linearly with distance.

Comparisons indicate that the pressure field decay is in very good agreement with theory. The agreement is not so good for the rise time, but the rate of increase with distance is correct: the creeping wave’s absorption results into long rise times in the deep shadow zone, of order 50 ms at 4 km. The computed rise time is probably underestimated because we have not taken into account (i) atmospheric turbulence and (ii) finite ground impedance. As shown previously, the influence of ground absorption on the finite rise time is expected to be significant near the cutoff. Taking into account that the rise time is extremely sensitive to the effects of even mild turbulence and the local conditions of the microphone emplacement, theoretical results can nevertheless be estimated satisfactory.

Figure 8 displays computed time signals at 0, 1, 2, 3, and 4 km from the carpet edge. We clearly observe the signal decay as it propagates deep into the shadow zone, and the rounding of the initial shock fronts because of diffraction-induced attenuation. The shock structure always spreads out after the initial shock front, as the creeping waves phase velocity is slightly smaller than the sound velocity. Figure 9 shows the normalized frequency spectrum (from 1 to 3000 Hz) of the signal at 1, 2, 3, and 4 km from the cutoff. It is obvious on the figure that the low-frequency spectrum remains unaltered during propagation, while the high-frequency content is strongly dampened by diffraction-induced attenuation. There is a 45-dB loss between 1 and 4 km at 1000 Hz, and a 70-dB loss at 3000 Hz. The maximum overpressure decay in the shadow zone is therefore due to the high-frequency attenuation, that spreads out the shock and erases the sharp pressure peak.

Figures 10–12 present the same results, but with computations over a grass-covered field (see Table II for the corresponding values). It must be noted that deep inside the shadow zone, the maximum overpressure and rise time curves (Fig. 10) are very similar to those of the rigid case. This can be explained as follows: according to the definition of parameter $q$, the influence of finite ground impedance is mostly sensitive at high frequencies. As the high frequencies are the most attenuated, they completely disappear deep into the shadow zone. Consequently, sufficiently far from the cutoff, there remain only low frequencies which propagate almost as if the ground surface was rigid. On the contrary, close to the carpet edge, the ground impedance influence is major, with an increase of the rise time and a decrease of the maximum overpressure. Compared to Concorde sonic booms, the rise time is in better agreement there now, but the maximum overpressure is slightly underestimated near the

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**FIG. 9.** Computed sonic boom pressure signals over a rigid surface. From left to right and top to bottom: normalized spectrum at 1, 2, 3, and 4 km from the carpet edge.

**FIG. 10.** Maximum overpressure and rise time in the shadow zone versus distance to the carpet edge over a grass-covered field. Theory (solid line) and Concorde measurements (○ and *).
cutoff. Maybe part of the initial rise was due to atmospheric turbulence.

Figure 11 shows the pressure time signals at the same distances as Fig. 8, but over a grass-covered field. The major difference with the rigid case is that time signals look more rounded near the cutoff. Figure 12 outlines the dramatic effect of finite ground impedance on the high-frequency content of the signal spectrum: there is now a 160 dB loss at 3000 Hz between 1 and 4 km. All frequencies above 100 Hz are strongly attenuated, and the sonic boom will only be perceived as a low-frequency rumbling noise.

VII. CONCLUSION

We have presented a geometrical theory of diffraction in a temperature-stratified atmosphere for predicting sonic booms in the shadow zone. A genuine extension to include horizontal winds was also proposed. Compared to previous studies, the main new points are (i) the proper matching to the nonlinear geometrical acoustics approximation over the carpet, (ii) the evaluation of nonlinear effects, (iii) the influence of finite ground impedance, and (iv) the comparison to in-flight (Concorde) data. Comparisons with Concorde measurements show the theory provides a reasonable evaluation of diffraction effects, though only the magnitude order of the rise time can be estimated. Numerical simulations outline the dramatic effects of atmospheric and ground conditions. Low temperature gradients or downwind propagation can lead to a large increase of the transition zone around the cutoff. This results in extremely long rise times at the cutoff.
as sound propagates over a finite impedance ground, even for a rather rigid ground. Further studies should take into account more realistic ground impedances (variable porosity or layered soils) and atmospheric turbulence.

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