An Introduction to Representation Theory of Finite Groups

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The groups discussed were mainly

- Symmetric groups
- $\mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/n\mathbb{Z})^*$
- $GL_n(\mathbb{C})$

Cayley (1894) gave definition of abstract group

(Definition) A set $G$ is called group if there exists an operation $\ast : G \times G \to G$ such that

1. $\ast$ is associative.
2. There exists an element $1 \in G$ such that $1a = a1 = a$ for all $a \in G$.
3. For every $a \in G$ there exists a unique $b \in G$ such that $ab = ba = 1$. 
Simple Groups - Group having no proper normal subgroups.

**Question**

*What are all the finite simple groups?*

To answer this question various tools were discussed and **Representation theory of finite groups** is one of these.

Other motivation of representation theory comes from the study of group actions.
Basic Definitions

$G$ - Always finite group.

**Definition**

A representation of $G$ is a homomorphism from $G$ to the set of automorphisms of a finite dimensional complex vector space $V$, i.e.

$$\phi: G \rightarrow GL(V)$$

$$\phi(g_1 g_2) = \phi(g_1) \phi(g_2).$$
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- $V$ is called a representation space for $\phi$. 
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- Dimension of $\phi := $ dimension of $V$. 
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- **Notation:** $(\phi, V)$ or $\phi$ or $V$. 
Basic Definitions

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**Definition**

A representation of $G$ is a homomorphism from $G$ to the set of automorphisms of a finite dimensional **complex** vector space $V$, i.e.

$$\phi: G \ni g \mapsto \phi(g) \in \text{GL}(V)$$

$$\phi(g_1g_2) = \phi(g_1)\phi(g_2).$$

- $V$ is called a representation space for $\phi$.
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- dimension of $\phi := \text{dimension of } V$.
- **Notation:** $(\phi, V)$ or $\phi$ or $V$.
- **Note:** For all $g \in G$, $\phi(g)V = V$. 

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Examples:

1. For any group $G$, the map $\phi : G \rightarrow \mathbb{C}^*$, given by $\phi(g) = 1$ for all $g$. $\dim(\phi) = 1$. 

2. Let $C_n$ be a cyclic group with $n$ elements. Let $\phi : C_n \rightarrow \mathbb{C}^*$ be a homomorphism, then $\phi(x)^n = 1$ for all $x \in C_n$.

3. Let $\omega$ be a generator of $C_n$ and $\zeta_n$ be the $n$th primitive root of 1. Then $\phi$ is completely determined by $\phi(\omega)$.

4. Let $\phi(\omega) = \zeta_i^n$ then we denote $\phi$ by $\phi_i$. 


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- Let \( C_n \) - Cyclic group with \( n \) elements. Let \( \phi : C_n \to \mathbb{C}^* \) a homomorphism, then
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Let \( \omega \) be a generator of \( C_n \) and \( \zeta_n \) be \( n \) \(-th\) primitive root of 1. Then \( \phi \) is completely determined by \( \phi(\omega) \).
Let \( \phi(\omega) = \zeta_n^i \) then we denote \( \phi \) by \( \phi_i \).
Examples

- Let $S_3$ - permutations of \{1, 2, 3\}. Define $\phi : S_3 \rightarrow \text{GL}_3(\mathbb{C})$ by

  
  \[
  (1) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
  (12) \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
  (13) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},
  (23) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},
  (123) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
  (132) \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
  
\]

- $\dim(\phi) = 3$. 

Examples

- (Permutation Representation) Suppose $G$ acts on finite set $X$, that is for each $s \in G$, there is given a permutation $x \mapsto sx$ of $X$ satisfying

$$1x = x, s(tx) = (st)x, \quad s, t \in G, x \in X.$$
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$$1x = x, s(tx) = (st)x, \quad s, t \in G, x \in X.$$ 

Let $V$ be complex vector space with basis $(e_x)_{x \in X}$. For $s \in G$, let

$$\rho : G \rightarrow \text{GL}(V);$$

$$\rho(s) : e_x \mapsto e_{sx}.$$ 

$$\dim(\rho) = |X|.$$
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- (Regular Representation) If $V$ is space with basis $(e_g)_{g \in G}$, then above action is called regular representation of $G$. 

Question

What are all the complex representations of a finite group $G$?
Definition

\textit{(G-invariant Space)} A space \( V \) is called \( G \)-invariant if there exists \( \phi : G \rightarrow \text{Aut}(V) \) such that

\[ \phi(g)V = V \quad \text{for all} \quad g \in G. \]
Definition

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- $G$-invariant $\leftrightarrow$ $G$-representation
- (Subrepresentation) Any $G$ invariant subspace of $V$ is called subrepresentation.
- (Irreducible Representation) A representation is called irreducible if it has no proper subrepresentations.
**Tools**

**Definition**

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- $G$-invariant $\iff$ $G$-representation
- (Subrepresentation) Any $G$ invariant subspace of $V$ is called subrepresentation.
- (Irreducible Representation) A representation is called irreducible if it has no proper subrepresentations.
- One dimensional representations are irreducible.
Tools

Definition

Two representations \((\phi_1, V_1)\) and \((\phi_2, V_2)\) are said to be equivalent if there exists an isomorphism \(T : V_1 \leftrightarrow V_2\) such that \(\phi_2(g)T = T\phi_1(g)\) for all \(g \in G\).
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V_1 \xrightarrow{\phi_1(g)} V_1 \\
\downarrow T \quad \downarrow T \\
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- In matrix notations: If \(\dim(V_1) = \dim(V_2) = n\). Then \((\phi_1, V_1) \cong (\phi_2, V_2)\), if there exists \(X \in \text{GL}_n(\mathbb{C})\) such that

\[
X\phi_1(g)X^{-1} = \phi_2(g).
\]
Examples

Consider $\phi_1 : S_2 \rightarrow GL_2(\mathbb{C})$ by

$$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

and $\phi_2 : S_2 \rightarrow GL_2(\mathbb{C})$ by

$$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (12) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
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Take $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then

$$X \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
Examples

- Consider one dimensional representations $\phi_i$ and $\phi_j$ of cyclic group $C_n$ given by

$$
\phi_i(\omega) = \zeta_n^i, \quad \phi_j(\omega) = \zeta_n^j
$$

where $\omega$ is generator of $C_n$. 

Claim: For $i \neq j$, $\phi_i \not\sim \phi_j$.

Proof: If $f: C \to C$ is an isomorphism then $f(x) = \lambda x$ for some $\lambda \in C^\times$. 

$C^\times \downarrow \downarrow \zeta_n^i \to \to C^\times \downarrow \downarrow \zeta_n^j \to \to C$ implies $\lambda \zeta_n^i = \zeta_n^j \lambda$, which is not true.
Examples

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**Claim:** For $i \neq j$, $\phi_i \not\cong \phi_j$.

**Proof:** If $f : \mathbb{C} \to \mathbb{C}$ is an isomorphism then $f(x) = \lambda x$ for some $\lambda \in \mathbb{C}^*$. 

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Definition

\((G\text{-linear map})\) Let \((\phi_1, V_1)\) and \((\phi_2, V_2)\) be two representations of finite group \(G\). Then a map \(T: V_1 \rightarrow V_2\) is called \(G\)-linear if

\begin{enumerate}
\item \(T\) is \(\mathbb{C}\)-linear.
\item \(T \circ \phi_1(g) = \phi_2(g) \circ T\).
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The kernel and image of \(G\)-linear map are \(G\)-invariant subspaces.
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- Let $W_1 \subset V_1$ be the kernel of $T$. 

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- \(T(\phi_1(g)v) = \phi_2(g)(T(v)) = 0\).
- \(\phi_1(g)v \in W_1\).
- Similar argument for the image.
(Direct Sum of Representations) If \((\phi, V)\) and \((\psi, W)\) be two representations of group \(G\), then \((\phi \oplus \psi, V \oplus W)\) given by

\[
[(\phi \oplus \psi)(g)](v, w) = (\phi(g)v, \psi(g)w)
\]

is a representation of \(G\).
More Tools

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Proposition

Let \((\phi, V)\) be a complex representation of finite group \(G\). The following are equivalent:

1. \((\phi, V)\) is irreducible.
2. \((\phi, V)\) can not be written as direct sum of two proper subrepresentations.

Proof:

Let \(W\) be a \(G\)-invariant subspace of \(V\). For proof we show that there is a complimentary invariant subspace \(W'\) such that \(V = W \oplus W'\). Let \(U\) be an arbitrary complement of \(W\) in \(V\), let \(\pi_0: V \rightarrow W\) be the projection given by the direct sum decomposition \(V = W \oplus U\).
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\[ V = W \oplus W'. \]

Let \(U\) be an arbitrary complement of \(W\) in \(V\), let

\[ \pi_0 : V \to W \]

be the projection given by the direct sum decomposition \(V = W \oplus U\).
Average the map $\pi_0$ over $G$, that is an onto map $\pi : V \rightarrow W$ by,

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} \phi(g)(\pi_0(\phi(g)^{-1}v)).$$

Then $\pi$ is a $G$-linear. Therefore its kernel is the required $G$-invariant complement of $W$. 
Theorem

*Every complex representation of finite group G is direct sum of irreducible representations.*
Theorem

Every complex representation of finite group $G$ is direct sum of irreducible representations.

Proof: (For cyclic group case) Let $\phi : C_n \rightarrow GL_m(\mathbb{C})$ - homomorphism.

Step 1. Every finite order complex matrix is diagonalizable.

Step 2. The matrices $\phi(x)$ are pairwise commuting and diagonalizable, hence are simultaneously diagonalizable. Therefore $\phi$ is easily seen to be direct sum of one dimensional representations. That is $\phi = \bigoplus_{i} \phi \oplus m_i$ where $m_i$ is the multiplicity.

General Proof

Decompose $V$ into irreducible representation by using last proposition.
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- **General Proof** Decompose $V$ into irreducible representation by using last proposition.
So the question is...

**Question**

*What are all the finite dimensional inequivalent irreducible complex representations of a given finite group G?*
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- Observe we have answered it already for cyclic groups.
Theorem

(Schur’s Lemma) Let $\phi_1 : G \to \text{GL}(V_1)$ and $\phi_2 : G \to \text{GL}(V_2)$ be two irreducible representations of $G$, and let $T$ be a linear mapping of $V_1$ into $V_2$ such that $\phi_2(g) \circ T = T \circ \phi_1(g)$ for all $g \in G$. Then:

1. If $\phi_1$ and $\phi_2$ are not isomorphic, we have $T = 0$.
2. If $V_1 = V_2$ and $\phi_1 = \phi_2$, then $T(x) = \lambda x$ for $x \in V$ and for some scalar $\lambda \in \mathbb{C}$. 

Proof:
Suppose $T \neq 0$. The $G$-linearity of $T$ implies that both kernel and image of $T$ are $G$-invariant subspaces.
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(Schur’s Lemma) Let $\phi_1 : G \to \text{GL}(V_1)$ and $\phi_2 : G \to \text{GL}(V_2)$ be two irreducible representations of $G$, and let $T$ be a linear mapping of $V_1$ into $V_2$ such that $\phi_2(g) \circ T = T \circ \phi_1(g)$ for all $g \in G$. Then:

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2. If $V_1 = V_2$ and $\phi_1 = \phi_2$, then $T(x) = \lambda x$ for $x \in V$ and for some scalar $\lambda \in \mathbb{C}$.

Proof:

Suppose $T \neq 0$. The $G$-linearity of $T$ implies that both kernel and image of $T$ are $G$-invariant subspaces.
More tools and Interesting Results

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**Theorem**

The number of inequivalent irreducible representations of finite group is equal to the number of its conjugacy classes.
Theorem

If $\rho_1, \rho_2, \ldots, \rho_t$ are all the inequivalent irreducible representations of group $G$ then

$$|G| = \sum_{i=1}^{t} \dim(\rho_i)^2.$$
More tools and Interesting Results

**Theorem**

*If* $\rho_1, \rho_2, \ldots, \rho_t$ *are all the inequivalent irreducible representations of group* $G$ *then*

$$|G| = \sum_{i=1}^{t} \dim(\rho_i)^2.$$ 

Consider the group $S_3$.

- $|S_3| = 6$.
- $S_3$ has three conjugacy classes given by $(1)$, $(12)$, $(123)$.
- Define $\sigma : S_3 \to \mathbb{C}^\ast$ by

  $$\sigma(1) = 1, \sigma(123) = \sigma(132) = 1,$$
  $$\sigma(12) = \sigma(23) = \sigma(13) = -1.$$
- $\sigma$ and trivial representations are two one dimensional inequivalent irreducible representations of $S_3$ of dimension one.
- The only way to write 6 as sum of three squares is $6 = 1 + 1 + 2^2$.
- Recall that the permutation representation of $S_3$ which maps each permutation to corresponding permutation matrices.
- This is not direct sum of one dimensional representations.
- The only decomposition possible is $3 = 1 + 2$.
- So its decomposition with above observations will give all the irreducible representations of $S_3$. 
Let $S_3$ - permutations of $\{1, 2, 3\}$. Define $\phi : S_3 \rightarrow \text{GL}_3(\mathbb{C})$ by

- $(1) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $(12) \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $(13) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $(23) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
- $(123) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $(132) \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

$\dim(\phi) = 3$. 
Set \( G = \text{GL}_2(K) \), \(|K| = q\) where \( q = p^r \) for an odd prime \( p \).

**Question**

*What are all the irreducible complex representations of \( G \)?*

We shall construct a very special class of representations of these groups.
Remark

A representation of dimension one of $G$ is simply a homomorphism $\phi$ of $G$ to $\mathbb{C}^*$. 

Let $G_0 = \{ xyx^{-1}y^{-1} | x, y \in G \}$ be the derived subgroup of $G$. Let $\pi: G \to G/G_0$ be the natural epimorphism.

Proposition (One dimensional representations of $G$) Let $\phi: G \to \mathbb{C}^*$ be a homomorphism. Then there exists a homomorphism $\bar{\phi}: G/G_0 \to \mathbb{C}^*$ such that $\bar{\phi}.\pi = \phi$. Conversely, any homomorphism from $G/G_0$ to $\mathbb{C}^*$ produces a homomorphism from $G$ to $\mathbb{C}^*$ by composition with $\pi$. 

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Definition

(Induced Representations) Let $H$ be a subgroup of a finite group $G$, and let $(\psi, U)$ be a representation of $H$. Let

$$V = \{f : G \to U | f(hg) = \psi(h)f(g), h \in H, g \in G\}.$$ 

Then $G$ acts on $V$ by right translations; $\phi : G \to \text{Aut}(V)$ by

$$[\phi(g)(f)](g') = f(g'g) \quad g, g' \in G, f \in V.$$ 

It follows that $(\phi, V)$ is a representation of $G$. This is called induced representation of $(\psi, U)$ from $H$ to $G$, denoted by $\text{Ind}_H^G(\psi)$. 

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- By restricting the action of $\phi$ on a subgroup $H$ of $G$, we obtain a representation of $H$, denoted by $\text{Res}_H^G(\phi)$.
Set $G = \text{GL}_2(K)$, $|K| = q$ where $q = p^r$ for an odd prime $p$.

**Question**

*What are all the irreducible complex representations of $G$?*

- Let $K^*$ be invertible elements of $K$.
- Recall $K^*$ is a cyclic group of order $q - 1$.
- Let $\hat{K}^*$ is the set of one dimensional representations of $K^*$.
- Let

$$B = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \in K^*, b \in K \}.$$  

- Choose $\mu_1, \mu_2 \in \hat{K}^*$.
  Define

$$\mu : B \mapsto \mathbb{C}^*; \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \mu_1(a) \mu_2(c).$$
Write

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Now define an action of \( G \) on \( V_\mu \) by

\[ (\hat{\mu}(g)f)(x) = f(xg), \quad \text{for all} \quad x, g \in G. \]

Dimension of \( \hat{\mu} = \frac{|G|}{|B|} = \frac{(q^2-1)(q^2-q)}{(q-1)^2q} = q + 1. \]
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Dimension of $\hat{\mu} = \frac{|G|}{|B|} = \frac{(q^2-1)(q^2-q)}{(q-1)^2q} = q + 1.$

**Question**

*Is $\hat{\mu}$ irreducible?*
Case 1. For $\mu_1 \neq \mu_2$, the $G$-representation $(\hat{\mu}, V_\mu)$ is irreducible. Furthermore,

$$\left(\mu_1, \mu_2\right) \cong \left(\mu'_1, \mu'_2\right)$$

if and only if either

$$\mu_1 = \mu'_1 \quad \text{and} \quad \mu_2 = \mu'_2$$

or

$$\mu_1 = \mu'_2 \quad \text{and} \quad \mu_2 = \mu'_1.$$

This gives $\frac{(q-1)(q-2)}{2}$ inequivalent $(q + 1)$ dimensional irreducible representations.
Case 2. For \( \mu_1 = \mu_2 \), the \( G \)-representation \((\hat{\mu}, V_\mu)\) is not irreducible. In this case

\[
\hat{\mu} = \phi_1 \oplus \phi_2,
\]

where \( \phi_1 \) is one dimensional and \( \phi_2 \) is \( q \) dimensional.

Suppose \( \hat{\mu} = \phi_1 \oplus \phi_2 \) and \( \hat{\nu} = \psi_1 \oplus \psi_2 \).

If \( \mu \not\equiv \nu \) then

\[
\phi_1 \not\equiv \psi_1 \quad \text{and} \quad \phi_2 \not\equiv \psi_2.
\]

This gives \((q - 1)\) one dimensional representations and \((q - 1)\) inequivalent \( q \) dimensional representations.
Table of Conjugacy classes

<table>
<thead>
<tr>
<th>conjugacy class</th>
<th>representative</th>
<th>No. of classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>central semisimple</td>
<td>((a \ 0))</td>
<td>(q - 1)</td>
</tr>
<tr>
<td>unitary</td>
<td>((a \ 1))</td>
<td>(q - 1)</td>
</tr>
<tr>
<td>non central semisimple</td>
<td>((a \ 0)), (a \neq b)</td>
<td>(\frac{(q-1)(q-2)}{2})</td>
</tr>
<tr>
<td>anisotropic</td>
<td>(0 \ -\alpha\bar{\alpha})\</td>
<td>(\frac{q^2-q}{2})</td>
</tr>
</tbody>
</table>
Mackey’s intertwining Theorem

- Let $H$ and $K$ be subgroups of $G$. If $g \in G$ then the set $HgK = \{ hgk \mid h \in H, k \in K \}$ is called a double coset with respect to subgroup $H$ and $K$. The element $g$ is called its representative.
- A complete set of representatives of all $(H, K)$-double cosets is denoted by $H \backslash G / K$.
- For $s \in H \backslash G / K$ we set $H_s = sHs^{-1} \cap K$.
- Consider the representation $(\tau, W)$ of $H$.
- The subgroup $H_s$ has a natural representation $(\tau^s, W)$ defined by

$$\tau^s(x) = \tau(s^{-1}xs), \quad x \in H_s$$
If $(\phi, V)$ and $(\psi, W)$ are two $G$-representations then

$$\text{Hom}_G(\phi, \psi) = \{ T : V \rightarrow W \mid T \text{ is } G \text{ - linear} \}.$$ 

**Theorem**

*(Mackey’s Intertwining Theorem)* Let $H$ and $K$ be subgroups of $G$ and $(U, \sigma)$ a representation of $K$ and $(W, \tau)$ a representation of $H$. Then:

$$\text{Hom}_G(\text{Ind}^G_K(\sigma), \text{Ind}^G_H(\tau)) \cong \bigoplus_{s \in H \setminus G/K} \text{Hom}_{sHs^{-1} \cap K}(\sigma, \tau^s).$$
Continuation for Cases I and II

For our case $H = K = B$, where $B = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, c, \in K^*, b \in K \}$
and $\sigma = \tau = (\mu_1, \mu_2)$.
Continuation for Cases I and II

- For our case $H = K = B$, where $B = \{(a \ b \ 0 \ c) \mid a, c, \in K^*, b \in K\}$ and $\sigma = \tau = (\mu_1, \mu_2)$.
- For $c \neq 0$:

$$
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} = \begin{pmatrix}
b - ac^{-1}d & a \\
0 & c \\
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} \begin{pmatrix}
1 & c^{-1}d \\
0 & 1 \\
\end{pmatrix}
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  \]
- The double coset representatives $B \backslash \text{GL}_2(K)/B$ correspond to
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  \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \},
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  \]
- Let $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then
  \[
  B \cap sBs^{-1} = T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a.b \in K^* \right\}.
  \]
Observe that

$$(\mu_1, \mu_2)^s \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \mu_2(x)\mu_1(y).$$
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\[(\mu_1, \mu_2) \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \mu_2(x)\mu_1(y).\]

Hence by Mackey’s intertwining theorem we obtain that

\[\text{Hom}_G(\widehat{(\mu_1, \mu_2)}, \widehat{(\mu_1, \mu_2)}) = [\text{Hom}_B((\mu_1, \mu_2), (\mu_1, \mu_2))] \oplus [\text{Hom}_T((\mu_1, \mu_2), (\mu_2, \mu_1))].\]

The $T$-representations $(\mu_1, \mu_2)$ and $(\mu_2, \mu_1)$ are equivalent if and only if $\mu_1 = \mu_2$.

This implies $\text{dim}_\mathbb{C}(\text{Hom}_G(\widehat{(\mu_1, \mu_2)}, \widehat{(\mu_1, \mu_2)}))$ is equal to one for $\mu_1 \neq \mu_2$ and is equal to two for $\mu_1 = \mu_2$.

This combined with Schur’s Lemma gives the result.
<table>
<thead>
<tr>
<th>dimension</th>
<th>No of irreducible representations</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$q - 1$</td>
<td>one dimensional</td>
</tr>
<tr>
<td>$q$</td>
<td>$q - 1$</td>
<td>special</td>
</tr>
<tr>
<td>$q + 1$</td>
<td>$\frac{(q-1)(q-2)}{2}$</td>
<td>regular principal series</td>
</tr>
<tr>
<td>$q - 1$</td>
<td>$\frac{q^2-q}{2}$</td>
<td>cuspidal</td>
</tr>
</tbody>
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References


