Solutions to Exercises 9

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1. $\mathcal{L}(\mathcal{P}_n; f) = \sum_{i=0}^{2} m_i(t_{i+1} - t_i)$, $\mathcal{U}(\mathcal{P}_n; f) = \sum_{i=0}^{2} M_i(t_{i+1} - t_i)$. Here, $t_0 = 0$, $t_1 = 1 - \frac{1}{n}$, $t_2 = 1 + \frac{1}{n}$, $t_3 = 2$ and $m_i = inf\{f(x) : x_i \leq x \leq x_{i+1}\}$ $M_i = sup\{f(x) : x_i \leq x \leq x_{i+1}\}$. $m_0 = M_1 = 1$, $m_1 = 1$, $M_1 = 2$, $M_2 = m_2 = 2$. So,
   $\mathcal{L}(\mathcal{P}_n) = (1 - \frac{1}{n}) + (1 + \frac{1}{n} - 1 + \frac{1}{n}) + 2(2 - 1 - \frac{1}{n}) = 3 - \frac{1}{n}$.
   $\mathcal{U}(\mathcal{P}_n) = (1 - \frac{1}{n}) + 2(1 + \frac{1}{n} - 1 + \frac{1}{n}) + 2(2 - 1 - \frac{1}{n}) = 3 + \frac{1}{n}$.
   $\mathcal{U}(\mathcal{P}_n; f) - \mathcal{L}(\mathcal{P}_n; f) = \frac{2}{n} < \epsilon$ if n is strictly grater than $\frac{2}{\epsilon}$.
   $\int_0^2 f(x) = sup\{\mathcal{L}(\mathcal{P}; f) : \mathcal{P}$ is a partition of $[0, 2]\}$=inf$\{\mathcal{U}(\mathcal{P}; f) : \mathcal{P}$ is a partition of $[0, 2]\} = 3$

2. To find the value of $\int_0^n [x]dx$.
   $\int_0^n [x]dx$
   $=\int_0^1 [x]dx + \int_1^2 [x]dx + ...+ \int_{n-1}^n [x]dx$
   $=\int_0^1 0dx + \int_1^2 1dx + ...+ \int_{n-1}^n (n-1)dx$
   $=0 + 1 + 2 + ... + (n-1)$
   $=\frac{(n-1)(n-2)}{2}$

3. To evaluate without doing any computations

   $$\int_{-1}^1 x^3\sqrt{1-x^2}dx$$

   $f(x) := x^3\sqrt{1-x^2}$ is an odd function so the value of the integration is 0.

4. Let us write $f(x) = x^5\sqrt{1-x^2}$ and $g(x) = 3\sqrt{1-x^2}$, then f is an odd function and g is an even function.

   $$\int_{-1}^{+1} f(x)dx = 0$$

   $\int_{-1}^{+1} 3\sqrt{1-x^2}dx$=$2\int_0^{+1} 3\sqrt{1-x^2}$
   $x = \sin\theta$
   $=2\int_0^{\frac{\pi}{2}} 3\sqrt{1-\sin\theta^2}\cos\theta d\theta$= $\int_0^{\frac{\pi}{2}} 6\cos^2\theta d\theta$
   $=3\int_0^{\frac{\pi}{2}} (1-\cos 2\theta)d\theta$
   $=\frac{3}{2}\pi$

5. To prove that $\int_0^x \frac{\sin t}{1+t}\,dt > 0$
Let $x = 2n\pi + x_0$. Then,
$\int_0^x \frac{\sin t}{1+t}\,dt = \sum_{i=0}^n \int_{2(i-1)\pi}^{2i\pi} \frac{\sin t}{1+t}\,dt + \int_{2n\pi}^{2n\pi+x_0} \frac{\sin t}{1+t}\,dt.$
Now, $\int_{2(i-1)\pi}^{2i\pi} \frac{\sin t}{1+t}\,dt = \int_{2(i-1)\pi}^{2i\pi-\pi} \frac{\sin t}{1+t}\,dt + \int_{2i\pi-\pi}^{2i\pi} \frac{\sin t}{1+t}\,dt$
$= \int_{2(i-1)\pi}^{2i\pi-\pi} \frac{\sin t}{1+t}\,dt - \int_{2(i-1)\pi}^{2i\pi-\pi} \frac{\sin t}{1+t+\pi}\,dt$
$=\int_{2(i-1)\pi}^{2i\pi-\pi} (\frac{\sin t}{1+t} - \frac{\sin t}{1+t+\pi})\,dt$
$=\int_{2(i-1)\pi}^{2i\pi-\pi} \sin t \frac{\pi}{(1+t)(1+t+\pi)}\,dt > 0$, since $\sin t \frac{\pi}{(1+t)(1+t+\pi)} > 0$ on $[2(i-1)\pi, 2i\pi-\pi]$.
Now look at the last term $\int_{2n\pi}^{2n\pi+x_0} \frac{\sin t}{1+t}\,dt.$ If $x_0 \in [0,\pi]$ then $\int_{2n\pi}^{2n\pi+x_0} \frac{\sin t}{1+t}\,dt > 0$ since $\frac{\sin t}{1+t} > 0$ on $[2n\pi, 2n\pi+x_0]$ for $x_0 \in [0,\pi]$.
Suppose $x_0 \in (\pi, 2\pi)$, then,
$\int_{2n\pi}^{2n\pi+x_0} \frac{\sin t}{1+t}\,dt$
$=\int_{2n\pi}^{2n\pi-\pi+x_0} \frac{\sin t}{1+t}\,dt + \int_{2n\pi-\pi+x_0}^{2n\pi+\pi} \frac{\sin t}{1+t}\,dt + \int_{2n\pi+\pi}^{2n\pi+x_0} \frac{\sin t}{1+t}\,dt$
$=\int_{2n\pi}^{2n\pi-\pi+x_0} (\frac{\sin t}{1+t} - \frac{\sin t}{1+t+\pi})\,dt + \int_{2n\pi-\pi+x_0}^{2n\pi+\pi} \frac{\sin t}{1+t}\,dt > 0$, since, $\frac{\sin t}{1+t} - \frac{\sin t}{1+t+\pi} > 0$ on $[2n\pi, 2n\pi-\pi+x_0]$ and $\frac{\sin t}{1+t} > 0$ on $[2n\pi-\pi+x_0, 2n\pi+\pi]$.
Hence, $\int_0^x \frac{\sin t}{1+t}\,dt > 0$ for every $x \in \mathbb{R}_+$.

6. Let $a < b < c < d$ and f is integrable on $[a,d]$. To prove that f is integrable on $[b,c]$.
Since f is integrable on $[a,d]$, for given $\epsilon > 0$ there exist a partition $\mathcal{P} = \{a = t_0 < t_1 < ..... < t_n = d\}$ so that, $\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon$.
Let us consider a refinement $\mathcal{P}'$ of $\mathcal{P}$ which includes b and c also. Then we have, $\mathcal{U}(\mathcal{P}', f) - \mathcal{L}(\mathcal{P}', f) < \epsilon$.
Suppose $\mathcal{P}' := a = s_0 < s_1 < ... < s_k = d$ and $s_{n_0} = b$ and $s_{n_1} = c$. Consider the partition $\mathcal{Q} := b = s_{n_0} < s_{n_0+1} < .... < s_{n_1} = c$ of $[b,c]$. Then it is easy to check that,

$$\mathcal{U}(\mathcal{Q}, f|_{[b,c]}) - \mathcal{L}(\mathcal{Q}, f|_{[b,c]}) < \epsilon.$$

Hence the function f is integrable on $[b,c]$.

7. Suppose f and g are two integrable functions on $[a,b]$ and $f(x) \geq g(x)$ for all x in $[a,b]$. To prove that,

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Define $h(x) := f(x) - g(x)$, then h is a non-negative integrable function on $[a,b]$ and it is enough to prove that, $\int_a^b h(x)\,dx \geq 0$. If h is non-negative, then its inf and *sup* on any subinterval in any partition are also non-negative. This implies that the upper sums $\mathcal{U}(h;\mathcal{P})$ and lower sums $\mathcal{L}(h;\mathcal{P})$ are non-negative for any partition $\mathcal{P}$. Since the integral is greater than or equal to every lower sum, it is

non-negative.So,
$\int_a^b h(x)\,dx \geq 0$
$\Rightarrow \int_a^b (f(x) - g(x))\,dx \geq 0$
$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8. To prove that,

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} g(x)\,dx,\ g(x) := f(x-c); \forall x \in [a+c, b+c].$$

If $\mathcal{P} := a = t_0 < t_1 < ... < t_n = b$ is a given partition of $[a,b]$ then for the partition $\mathcal{P}' := a + c = t_0 + c < t_1 + c < ... < t_n + c = b + c$ of $[a+c, b+c]$ we have, $\mathcal{U}(f;\mathcal{P}) = \mathcal{U}(g;\mathcal{P}')$ and $\mathcal{L}(f;\mathcal{P}) = \mathcal{L}(g;\mathcal{P}')$.
On the other hand, for a given partition $\mathcal{Q} := a + c = q_0 < q_1 < .... < q_m = b + c$ of $[a+c, b+c]$ there is a partition $\mathcal{Q}' := a = q_0 - c < q_1 - c < .... < q_m - c = b$ of $[a,b]$ so that, $\mathcal{U}(f;\mathcal{Q}) = \mathcal{U}(g;\mathcal{Q}')$ and $\mathcal{L}(f;\mathcal{Q}) = \mathcal{L}(g;\mathcal{Q}')$. Hence we have,

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} g(x)\,dx.$$

9. To find the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = \frac{x^2}{2} + 2$.
$f(x) = g(x) \Leftrightarrow x = +2, -2$; and on $[-2, +2]$ $g(x) \geq f(x)$. Hence the area is given by,
$\int_{-2}^{+2} (g(x) - f(x))\,dx$
$= \frac{16}{3}$.(check it.)

10. To find the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = 1 - x^2$.
$f(x) = g(x) \Leftrightarrow x^2 = \frac{1}{2} \Leftrightarrow x = +\frac{1}{\sqrt{2}}, -\frac{1}{sqrt2}$ and $g(x) \geq f(x)$ on the interval $(-\frac{1}{\sqrt{2}}, \frac{1}{sqrt2})$. Hence the area is given by,
$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (g(x) - f(x))\,dx = \frac{5}{3\sqrt{2}}$.(check it!)