Solutions to Exercises 8
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1. Let $f(x) = \sin(x)$, then $P_{3,0}(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$. Hence $P_{3,0}(x) = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-1) \Rightarrow P_{3,0}(x) = x - \frac{x^3}{6}$.
$x^2 = \sin(x) \Rightarrow x^2 = x - \frac{x^3}{6}$. It is easy to see that r is a non zero root of the equation $x^2 = x - \frac{x^3}{6} \Leftrightarrow x(x^2 + 6x - 6) = 0$

2. See n-th derivative of $f + g$, $(f + g)^n(x) = f^n(x) + g^n(x)$. Given $P_{n,a}(x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^n}{n!}f^{(n)}(a)$ and $Q_{n,a}(x) = g(a) + xg'(a) + \frac{x^2}{2!}g''(a) + \frac{x^3}{3!}g'''(a) + \cdots + \frac{x^n}{n!}g^{(n)}(a)$. Let Taylor's Polynomial for $f + g$ is $S_{n,a}(x)$. Then $S_{n,a}(x) = (f + g)(a) + x(f + g)'(a) + \frac{x^2}{2!}(f + g)''(a) + \frac{x^3}{3!}(f + g)'''(a) + \cdots + \frac{x^n}{n!}(f + g)^{(n)}(a) = P_{n,a}(x) + Q_{n,a}(x)$.
If $f^{(n)}$ and $g^{(n)}$ exists then using Leibniz's formula we have
$(f \cdot g)^{(n)}(a) = \Sigma_{k=0}^{n}\binom{n}{k}f^{(k)}(a)g^{(n-k)}(a)$
Now Taylor's Polynomial for $f(x)g(x)$ is $= (f \cdot g)(a) + x(f \cdot g)'(a) + \frac{x^2}{2!}(f \cdot g)''(a) + \frac{x^3}{3!}(f \cdot g)'''(a) + \cdots + \frac{x^3}{3!}(f \cdot g)^{(n)}(a)$.
Now using Leibniz's formula we can find the Taylor's Polynomial for $f(x)g(x)$.

3. In each case, let $f(x) = y$ then $x = f^{-1}(y)$.
   (a) $f^{-1}(y) = y$ if $y \in \mathbb{Q}$ and $f^{-1}(y) = -y$ if $y \in \mathbb{R} - \mathbb{Q}$.
   (b) $y = f(x) = x + n$ if $n \le x < n + 1 \Rightarrow f^{-1}(y) = y - n$ if $2n \le y < 2n + 1$ for all $n \in \mathbb{Z}$.
   (c) $y = \frac{x}{1-x^2} \Rightarrow y \cdot x^2 + x - y = 0 \Rightarrow f^{-1}(y) = x = \frac{-1+\sqrt{1-4y^2}}{2y}$ since $-1 < x < 1$.

4. Given f is an increasing function. Let $z = f(x)$ and $w = f(y) \Leftrightarrow x = f^{-1}(z)$ and $y = f^{-1}(w)$. Let $z < w$. Now if $x = y$ then $f(x) = f(y)$ (since f is a function) $\Leftrightarrow z = w$. If $x > y$ then $f(x) > f(y) \Leftrightarrow z > w$, since f is an increasing function. In both case contradiction arrise since we have $z < w$. Hence we must have $x < y \Leftrightarrow f^{-1}(z) < f^{-1}(w)$. Hence f^{-1} is also a increasing function. Similar proof for decreasing one.

5. Let $fog(x_1) = fog(x_2) \Rightarrow f(g(x_1)) = f(g(x_2)) \Rightarrow g(x_1) = g(x_2)$, since f is one-one $\Rightarrow x_1 = x_2$, since g is one-one. Hence fog is one-one.
Let $(fog)^{-1}(z) = x \Rightarrow fog(x) = z \Rightarrow f(g(x)) = z \Rightarrow g(x) = f^{-1}(z) \Rightarrow x = g^{-1}(f^{-1}(z)) \Rightarrow x = (g^{-1}of^{-1})(z)$. Hence $(fog)^{-1}(z) = (g^{-1}of^{-1})(z)$.

6. Let $f^{-1}(y) = x \Rightarrow f(x) = y$. Then $g(x) = 1 + y \Rightarrow g^{-1}(1 + y) = x = f^{-1}(y)$. Replacing y by $y - 1$, we have $g^{-1}(y) = f^{-1}(y - 1)$.

7. Claim, f is one-one $\Rightarrow ad - bc \neq 0$.
   Let if possible $ad - bc = 0 \Rightarrow ad = bc$ but $(d, c) \neq (0, 0)$. If $a = 0$, then b or c is zero. Then $f(x) = 0$ of $\frac{b}{d}$. In both cases f is not one-one. Hence $a \neq 0$, then $f(x) = \frac{ax+b}{cx+d} = \frac{a^2x+ab}{acx+ad} = \frac{a^2x+ab}{acx+bc} = \frac{a}{c}$ (if $c = 0 \Rightarrow ad = 0 \Rightarrow d = 0$ but we have $(d, c) \neq (0, 0)$, hence $c \neq 0$). Also in this case, f is not one-one. Hence we must have $ad - bc \neq 0$.
   Claim, $ad - bc \neq 0 \Rightarrow f$ is one-one.
   Let $ad - bc > 0$. See $f'(x) = \frac{ad-bc}{(cx+d)^2}$. So, if $ad - bc > 0$ then $f'(x) > 0 \ \forall \ x$. For any a, b $(a < b)$ we have $f(b) - f(a) = f'(c)(b - a) > 0$ for some $c \in (a, b)$. Hence $a < b \Rightarrow f(a) < f(b) \Rightarrow$ f is an increasing function (in particular f is one-one).
   Similarly, if $ad - bc < 0$ then f is a decreasing function (in particular f is one-one). Hence done.

8. Let $f(x) = x^3 - 3x^2 \Rightarrow f'(x) = 3x^2 - 6x = 3x(x - 2)$.
   If $x < 0$ or $2 < x < \infty$, then $f'(x) > 0$. Hence in those intervals f is an increasing function, i.e. f is one-one on $[a, b]$ where $a, b \in (-\infty, 0]$ or $a, b \in [2, \infty)$.
   If $0 < x < 2$, then $f'(x) < 0$. Hence in that interval f is a decreasing function, i.e. f is one-one on $[a, b]$ where $a, b \in [0, 2]$.

9. $fof^{-1}(x) = x \Rightarrow f(f^{-1}(x)) = x \Rightarrow f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1 \Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. Now given $g = f^{-1}$ and $f'(x) = \frac{1}{\sqrt{1+x^3}}$. So $g'(x) = \frac{1}{f'(g(x))} = \sqrt{1 + g^3(x)}$. Hence $g''(x) = \frac{1}{2}(1 + g^3(x))^{-\frac{1}{2}} \cdot 3g^2(x) \cdot g'(x) = \frac{1}{2}\frac{1}{g'(x)} \cdot 3g^2(x) \cdot g'(x) = \frac{3}{2}g^2(x)$.

10. Since f is one-one, f^{-1} is a function whose inverse is f.
    Since f^{-1} is differentiable, f^{-1} is continuous.
    Let $y = f(x)$, so $f^{-1}(y) = x$.
    $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-y}{h}$.
    Now since f^{-1} is continuous, we can write $x + h = f^{-1}(y + k)$ where $k \to 0$ as $h \to 0$.
    Now $\lim_{h\to 0} \frac{f(x+h)-y}{h} = \lim_{h\to 0} \frac{f(f^{-1}(y+k))-y}{(x+h)-x}$
    $= \lim_{h\to 0} \frac{(y+k)-y}{f^{-1}(y+k)-f^{-1}(y)} = \lim_{k\to 0} \frac{k}{f^{-1}(y+k)-f^{-1}(y)}$ as "$h \to 0 \Rightarrow k \to 0$".
    Hence $\lim_{h\to 0} \frac{f(x+h)-y}{h} = \lim_{k\to 0} \frac{k}{f^{-1}(y+k)-f^{-1}(y)}$
    $= \frac{1}{(f^{-1})'(y)} = \frac{1}{(f^{-1})'(f(x))}$. So f is differentiable.