Solutions to Exercises 7

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1. Let the quadratic polynomial be $p(x) = a_0 + a_1 x + a_2 x^2$. We need to show that the slope of the chord joining the points $(a, p(a))$ and $(b, p(b))$ is equal to the slope of the tangent to $p(x)$ at the point $\frac{a+b}{2}$.
   Now the slope of the chord is $= \frac{p(b)-p(a)}{b-a} = \frac{a_1(b-a)+a_2(b^2-a^2)}{b-a} = a_1 + a_2(b+a)$.
   Slope of the tangent at the point $\frac{a+b}{2}$ is $= p'(\frac{a+b}{2}) = a_1 + 2a_2\frac{a+b}{2}$ $= a_1 + a_2(b+a)$. Hence the proof.

2. Suppose there are two points x_1 and x_2 belonging to $[-1, 1]$ such that $x^3 - 3x + b = 0$ for $x = x_1$ and $x = x_2$. Then by Rolle's theorem there exist a $c \in (-1, 1)$ (in particular in $(x_1, x_2) \subset (-1, 1)$) such that $3c^2 - 3c = 0$. But by solving $3c^2 - 3c = 0$ we get $c = \pm 1$. Hence we arrive at a contradiction. Thus there can exist at most one $x \in [-1, 1]$ such that $x^3 - 3x + b = 0$.

3. Let $f(x) = x^2 - x \sin x - \cos x$, then $f(\frac{\pi}{4}) < 0$ (check) and $f(\pi) > 0$. So by intermediate value property there exist a real root of $f(x)$ in $(\frac{\pi}{4}, \pi)$. Again $f(0) < 0$ and $f(-\pi) > 0$. So again by intermediate value property there exists a real root of $f(x)$ in $(-\pi, 0)$. So there exists two real roots of $f(x)$. Now suppose that there are 3 or more real roots of the equation $f(x)$. Then by Rolle's theorem there exist two distinct points x_1 and x_2 which are roots of the equation $f'(x) = 2x - x\cos x = 0$. But the equation $2x = x \cos x$ has only one real solution given by $x = 0$. Thus $f(x)$ has exactly two roots.

4. Try it yourself.

5. Now $f(x) \geq 24$ means $A \geq (24 - 5x^2)x^5$. So let $g(x) = (24 - 5x^2)x^5$ then A should be greater then the maximum of $g(x)$ whenever $x > 0$. So the minimum value of A, such that $f(x) \geq 24$ holds is maximum of $\{g(x) : x > 0\}$. Now $g'(x) = 5x^4(24 - 7x^2)$ and $g''(x) = 20x^3(24 - 7x^2) + 5x^4(24 - 14x)$. Note that the only possible positive root of $g'(x)$ is at $x = \sqrt{\frac{24}{7}}$ and $g''(\sqrt{\frac{24}{7}}) < 0$(?check), hence it is the maximum. So $A = g(\sqrt{\frac{24}{7}})$.

6. We have $f'(x) = \sum_{i=1}^{n} 2(x - a_i)$. Thus if $x = \frac{a_1+a_2+\cdots+a_n}{n}$ then $f'(x) = 0$.

Also $f''(x) = n$. Hence the minimum is attained at the arithmetic mean.

7. Let $S = 1, 1/2, 1/3, ...$ then the function is defined as $f(x) = 1$ on S and $f(x) = 0$ on $\mathbb{R} \setminus S$. So on any neighbourhood of $x \in S \cup \{0\}$ the local maxima is 1 and the local minima is 0. And around any other point $x \in \mathbb{R} \setminus (S \cup \{0\})$ there exists sufficiently small interval such that (say I_x) $I_x \cap S = \emptyset$. So the both the local maxima and the local minima is 0.

8. (a) By MVT we have $f(b) - f(a) = f'(c)(b - a)$ for some $c \in (a, b)$. Since $f'(x) \geq M$ for all $x \in [a, b]$ we get

$$f(b) \geq f(a) + M(b - a)\,.$$

(b) Since $f'(x) \leq M$ for all $x \in [a, b]$ we get

$$f(b) \leq f(a) + M(b - a)\,.$$

(c) Taking modulus on both sides of the MVT we have

$$|f(b) - f(a)| = |f'(c)|(b - a)\,.$$

Now if $|f'(x)| \leq M$ for all $x \in [a, b]$, then we get

$$|f(b) - f(a)| \leq M(b - a)\,.$$

9. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \frac{a_0 x}{1} + \frac{a_1 x^2}{2} + \cdots + \frac{a_n x^{n+1}}{n+1}\,.$$

Then by the given hypothesis $f(0) = f(1) = 0$. So by Rolle's theorem there exists some $x \in (0, 1)$ such that $f'(x) = 0$. Now observe that $f'(x) = a_0 + a_1 x + \cdots + a_n x^n$. Hence the proof follows.

10. Assume that f has zero at two points say c_1 and c_2 such that $f(x) \neq 0$ for all $x \in (c_1, c_2)$. So on the compact set $[c_1, c_2]$ f will have a a maxima and a minima. Since $f(x) \neq 0$ for all $x \in (c_1, c_2)$ then by IVP either $f(x) \geq 0$ $\forall x \in [c_1, c_2]$ or $f(x) \leq 0$ $\forall x \in [c_1, c_2]$. So if $f(x) \leq 0$ $\forall x \in [c_1, c_2]$ then clearly the maxima is the point c_1 and c_2. So let $c \in (c_1, c_2)$ be the minima then $f'(c) = 0$ i.e. $f''(c) - f(c) = 0$ or $f''(c) < 0$ which is a contradiction to the fact that c is minima. Hence the minima on $[c_1, c_2]$ is at c_1 and c_2. Which means f is 0 on $[c_1, c_2]$. Similarly prove for the case when $f(x) \geq 0$ $\forall x \in [c_1, c_2]$.