Solution to Exercise – 6

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1. We know that $f(x) = [x]$ is a continuous function except at integers. So f is not differentiable at integers. For non-integers the derivative is zero. Since for any $a \in \mathbb{R}$ which is not an integer, there exists a unique n in $\mathbb{Z}$ such that

$$n < a < n+1 \text{ and } f(a) = n.$$

We can find $h > 0$ such that $a + h \in (n, n+1)$ and $a - h \in (n, n+1)$, so

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{n-n}{h} = 0.$$

Similarly

$$\lim_{h\to 0} \frac{f(a-h) - f(a)}{h} = 0.$$

2. Since f is a periodic function with period a, for any x we have

$$f^{'}(x+a) = \lim_{h\to 0} \frac{f(x+a+h) - f(x+a)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f^{'}(x).$$

Hence $f^{'}$ is also a periodic function with period a.

3. Given that $f^{'}(a-) = g^{'}(a+)$. Now for any $x > 0$ we have

$$h^{'}(a-) = \lim_{x\to 0} \frac{h(a-x) - h(a)}{-x} = \lim_{x\to 0} \frac{f(a-x) - f(a)}{-x} = f^{'}(a-)$$

and

$$h^{'}(a+) = \lim_{x\to 0} \frac{h(a+x) - h(a)}{x} = \lim_{x\to 0} \frac{g(a+x) - g(a)}{x} = g^{'}(a+).$$

Now by the given condition we have

$$h^{'}(a+) = h^{'}(a-) = h^{'}(a).$$

4. $\left|\frac{f(0+h)-f(0)}{h} - 0\right| = \left|\frac{f(h)-0}{h}\right| = \left|\frac{f(h)}{h}\right| \le \left|\frac{h^2}{h}\right| = |h| \to 0$ as $h \to 0$.

This shows that f is differentiable at 0 and derivative is

$$f^{'}(0) = 0.$$

5. Given that $|f(x)| \le |x|^{1+\delta}$, so $f(0) = 0$. Now

$$\left|\frac{f(0+h) - f(0)}{h} - 0\right| = \left|\frac{f(h) - 0}{h}\right| = \left|\frac{f(h)}{h}\right| \le \frac{|h|^{1+\delta}}{|h|} = |h|^{\delta} \to 0 \text{ as } h \to 0.$$

This shows that f is differentiable at 0 and derivative is

$$f^{'}(0) = 0.$$

6. Define

$$h(x) := f(x) - g(x).$$

Then $h(x) = 0$ for all x in some open interval containing a. That is there exists $b > 0$ such that $h(x) = 0$ for all $x \in (a-b, a+b)$. Now for all x such that $|x - a| < b$ we have

$$\frac{h(a+x) - h(a)}{x} = \frac{0-0}{x} = 0.$$

That is

$$h^{'}(a) = \lim_{x \to 0} \frac{h(a+x) - h(a)}{x} = 0, \text{ that is, } f^{'}(a) = g^{'}(a).$$

7. Look at, $\left|\frac{f(0+h)-f(0)}{h} - 0\right| = \left|\frac{g(h)sin(\frac{1}{h})-f(0)}{h}\right| = \left|\frac{g(h)sin(\frac{1}{h})-0}{h}\right| = \left|\frac{g(h)\sin(\frac{1}{h})}{h}\right|$. That is

$$\left|\frac{f(0+h) - f(0)}{h} - 0\right| \leq \left|\frac{g(h)}{h}\right| = \left|\frac{g(h) - g(0)}{h}\right|.$$

This shows that

$$\lim_{h \to 0} \left|\frac{f(0+h) - f(0)}{h} - 0\right| \leq \lim_{h \to 0} \left|\frac{g(h) - g(0)}{h} - g^{'}(0)\right| = 0.$$

That is f differentiable at 0 and derivative is

$$f^{'}(0) = 0.$$

8. (a) Given that f is differentiable at a and $f(a) \neq 0$. Since f is continuous at a and therefore there exists some $\delta > 0$ such that for all $x \in I := (a - \delta, a + \delta)$, f has the same sign at x as it does at a. That is , if $f(a) > 0$ then $f(x)$ is positive for all $x \in I$ and if $f(a) < 0$ then $f(x)$ is negative for all $x \in I$. Suppose that $f(a) > 0$ then $|f| = f$ on I and hence $|f|$ is differentiable as f is differentiable. Similarly if $f(a) < 0$ then $|f| = -f$ on I and hence $|f|$ is differentiable as $-f$ is differentiable.

   (b) Define

   $$f(x) := x.$$

   Try to prove that $|f|$ is not differentiable at 0. Note that here $f(0) = 0$.

   (c) Given that f and g are differentiable function at a and $f(a) \neq g(a)$. We know that

   $$\max(f, g) = \frac{|f-g| + f + g}{2} \text{ and } \min(f, g) = \frac{f + g - |f-g|}{2}.$$

   Now by using part (a) we have $|f - g|$ is differentiable at a and we know that the sum and the difference of two differentiable functions are also differentiable. Hence we have the required result.

9. Use induction on n to get the desired result.

10. Given that $f(x) = xg(x)$ and g is continuous at 0. Now $\frac{f(0+h)-f(0)}{h} = \frac{hg(h)-0}{h} = g(h)$. Now by taking the limit and using the continuity of g we have

$$\lim_{h\to 0}[\frac{f(0+h)-f(0)}{h} - g(0)] = \lim_{h\to 0}[g(h) - g(0)] = 0.$$

That is

$$f^{'}(0) = g(0).$$