Solution to Exercises – 5

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1. The function $f : [a, b] \to \mathbb{R}$ is continuous that takes only rational values. **Claim:** f is constant. If not suppose f takes two distinct rationals x and y. But in between two rationals we can always get an irrational number, say z. But by Intermediate Value Theorem(IVT) there exists a $t \in (x, y)$ such that $f(t) = z$. This contradicts the hypothesis.

2. Consider the function $f = \sin x - x + 1$ on $[0, \pi]$. Then $f(0) > 0$ and $f(\pi) < 0$. By IVT we have the desired conclusion.

3. Consider the function $g(x) = x^{179} + \frac{163}{1+x^2+\sin^2 x} - 119$. Note that this function is continuous on $\mathbb{R}$. Now observe that $\lim_{x\to-\infty} g(x) = -\infty$. To see this, note that g contains three parts: the first part $x^{179} \to -\infty$ as $x \to -\infty$, for 179 is odd, the second part $\frac{163}{1+x^2+\sin^2 x} \to 0$ as $x \to -\infty$, for the denominator goes to ∞.

   Similarly argue that $\lim_{x\to\infty} g(x) = \infty$. But this says that if you choose x sufficiently large then $g(x) > 0$ and if you choose x sufficiently small and negative then $g(x) < 0$. As g is continuous , by IVT there exists a c such that $g(c) = 0$,. Hence we have the desired conclusion.

4. We show that in fact on $(-1, 1)$ either $f(x) = \sqrt{1 - x^2}$ or $-\sqrt{1 - x^2}$, $\forall x$. For if this is not true then there exist $c, d \in (-1, 1)$, $c \neq d$ such that $f(c) = \sqrt{1 - c^2}$ and $f(d) = -\sqrt{1 - d^2}$ or vice-versa but then $f(c)$, $f(d)$ have opposite sign. As f is continuous, by IVT f has to attain zero somewhere in between c and d. In particular this says that f has a zero, which is not 1 or -1. But this is a contradiction. Hence we have that either $f(x) = \sqrt{1 - x^2}$ or $-\sqrt{1 - x^2}$, $\forall x \in (-1, 1)$. Since $f(1) = f(-1) = 0$, we have the desired conclusion.

5. Consider the function $h = g/f$. By the assumption $h^2 = 1$. Therefore proceeding in the same manner as in the previous problem we conclude that either $h(x) = 1$ for all x or $h(x) = -1$ for all x. This gives the required conclusion.

6. Consider the function $h(x) = g(x) - f(x)$ on [a,b]. Now by assumption we have $h(a) > 0$ and $h(b) < 0$. Thus by IVT there is an $x \in (a, b)$ such that $h(x) = 0$.

7. Consider the function $h(x) = f(x) - x$ on [a,b]. Since the range of f is [a,b], f can have maximum value b and minimum value a. Thus $h(a) \leq 0$ and $h(b) \geq 0$. Again by IVT the required conclusion follows.

8. Since f is defined on a closed, bounded interval it must be bounded. Therefore it has an infimum, i.e., there is a number $z \geq 0$ in $\mathbb{R}$ such that $f(x) \geq z$ for all x. If $z = 0$ then by the infimum property there is a sequence of points from the image (say $y_n = f(x_n)$) such that $y_n \to 0$. Now take the sequence x_n in [a,b]. By Bolzano-Weierstrass theorem it has a convergent subsequence $x_{n_k} \to x_0$(say) in [a,b]. Now by

continuity $f(x_{n_k}) \to f(x_0)$. But since $f(x_n)$ converges to zero, the subsequence has to go to zero and hence $f(x_0) = 0$.. This contradicts the assumption that $f(x) > 0$ for all x.

9. Fix an x_0. For this x_0 there is a point y_0(say) such that $|f(y_0)| \leq \frac{1}{2}|f(x_0)|$. Now for this y_0 there is a point y_1(say) such that $|f(y_1)| \leq \frac{1}{2}|f(y_0)$. Continuing this way we get a sequence of points $(y_0, y_1, ..., y_n, ...)$ such that $|f(y_n)| \leq \frac{1}{2^{n+1}}|f(x_0)|$. Now since $f(x_0)$ is a fixed number and $\frac{1}{2^n}$ is a convergent sequence which goes to 0, so $f(y_n) \to 0$. Again by Bolzano-Weierstrass theorem (y_n) has a convergent subsequence $(y_{n_k}) \to z$(say). By the continuity $f(y_{n_k}) \to f(z)$. But this subsequence goes to zero and hence $f(z) = 0$.

10. Consider the function $g(x) = f(x + \frac{1}{2}) - f(x)$ on $[0, \frac{1}{2}]$. Then $g(0) = f(\frac{1}{2}) - f(0)$ and $g(\frac{1}{2}) = f(1) - f(\frac{1}{2})$. But by the hypothesis this gives $g(0) = -g(\frac{1}{2})$. Then if $g(0) = 0$, 0 is the required point . If not then $g(0)$ and $g(\frac{1}{2})$ have opposite signs. Thus by IVT the required conclusion follows.