Solution to Exercises – 4

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(1) Consider the following function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by,

$$f(x) = \begin{cases} -1 & , x \in \mathbb{Q} \\ +1 & , x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers. Then it is easy to check that f is continuous nowhere and $|f|(x) = 1$ for all $x \in \mathbb{R}$, is continuous everywhere.

(2) Let us consider two functions f and g such that,
· g is continuous at 0,
· $g(0) = 0$ and
· $|f(x| \leq |g(x)|, \forall x \in \mathbb{R}$.
To show that f is continuous at 0.
Let $\epsilon > 0$. Since g is continuous at 0, there exists a $\delta > 0$ such that, $|g(x) - g(0)| < \epsilon$ for all x satisfying $|x| < \delta$.
Now $|f(0) \leq |g(0)| = 0 \Rightarrow f(0) = 0$.

$$|f(x) - f(0)| = |f(x)| < |g(x)|$$
$$\Rightarrow |f(x) - f(0)| < \epsilon, \forall |x| < \delta.$$

Hence f is continuous at 0.

(3) Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by,

$$f(x) = \begin{cases} x & , x \in \mathbb{Q} \\ -x & , \text{otherwise.} \end{cases}$$

This function is continuous only at the point 0 (why?).
First take $x_0 \neq 0$, and claim that the function is not continuous at x_0. Since if $x_0 \neq 0$ then by taking a sequence of rational numbers converging to x_0 and then taking a sequence of irrational number converging to x_0 one can see that $lim_{x \to x_0} f(x)$ does not exist.
Now we will show that f is continuous at 0. We know that $f(0) = 0$ from the definition of f. We also know that $-|x| \leq f(x) \leq |x|$ for all x. Since $lim_{x \to 0}|x| = lim_{x \to 0} - |x| = 0$ it follows from the sandwich theorem that $lim_{x \to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0.
Now for given $a \in \mathbb{R}$ the function F defined by,

$$F(x) = f(x - a) \ , \ x \in \mathbb{R}$$

is continuous at a but not continuous elsewhere.

(4) Define, $f : \mathbb{R} \longrightarrow \mathbb{R}$ by,

$$f(x) = \begin{cases} -1 & ; x \in \mathbb{R} \setminus \{\frac{1}{n} | n \in \mathbb{N}\} \\ \frac{1}{n} - 1 & ; x = \frac{1}{n}, n \in \mathbb{N} \end{cases}$$

It is easy to see that f is continuous on $\mathbb{R} \setminus \{\frac{1}{n} | n \in \mathbb{N}\}$ and that it is discontinuous on $\{\frac{1}{n} | n \in \mathbb{N}\}$.

(5) Define, $f : \mathbb{R} \longrightarrow \mathbb{R}$ by,

$$f(x) = \begin{cases} -1 & ; 0 \neq x \in \mathbb{R} \setminus \{\frac{1}{n} | n \in \mathbb{N}\} \\ n & ; x = \frac{1}{n}, n \in \mathbb{N} \\ 0 & ; x = 0. \end{cases}$$

It is easy to check that f is discontinuous at $1, \frac{1}{2}, \frac{1}{3}$...... and at 0 but continuous at all other points.

(6) Suppose that the f satisfies that $f(x + y) = f(x) + f(y)$ for all x, y and f is continuous at 0. To show that f is continuous everywhere.

$$f(x) = f(x - y + y) = f(x - y) + f(y)$$
$$\Rightarrow f(x - y) = f(x) - f(y) \text{ and } f(0) = 0.$$

Let x_0 be an arbitrary point in $\mathbb{R}$. Since f is continuous at 0 for given $\epsilon > 0$ there exist $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon; \text{ whenever } |x| < \delta$$
$$\Rightarrow |f(x)| < \epsilon; \text{ whenever } |x| < \delta$$
$$\Rightarrow |f(x - x_0)| < \epsilon, \text{ whenever } |x - x_0| < \delta$$
$$\Rightarrow |f(x) - f(x_0)| < \epsilon; \text{ whenever } |x - x_0| < \delta.$$

Hence the function f is continuous at x_0 and the point x_0 is an arbitrary point in $\mathbb{R}$, so f is continuous everywhere.

(7) Suppose that f is continuous at a and $f(a) = 0$ and α is nonzero real number. To show that $f + \alpha$ is nonzero in some open interval containing a.

Since f is continuous at a, for given $\epsilon = |\frac{\alpha}{2}| > 0$ there exist a $\delta > 0$ such that,

$$|f(x) - f(a)| < \epsilon; \text{ if } |x - a| < \delta$$
$$|f(x)| < \epsilon \text{ if } |x - a| < \delta.$$

Now,

$$|f(x) + \alpha| > |\alpha| - |f(x)|, \text{ if } |x - a| < \delta$$
$$\Rightarrow |f(x) + \alpha| > |\alpha| - \epsilon, \text{ if } -\delta + a < |x| < \delta + a$$
$$\Rightarrow |f(x) + \alpha| > |\alpha| - \frac{|\alpha|}{2}, \text{ if } -\delta + a < x < \delta + a$$

Hence $|f+\alpha| > \frac{|\alpha|}{2} > 0$ on $(-\delta+a, \delta+a)$, i.e. $f+\alpha$ is nonzero in $(-\delta+a, \delta+a)$ which is what we want to prove.

(8) (i) Suppose f is continuous at a. To prove that $|f|$ is continuous at a. Since f is continuous at a, for given $\epsilon > 0$ there exist $\delta > 0$ such that,

$$|f(x) - f(a)| < \epsilon, \text{ when } |x-a| < \delta$$

Now for given ϵ we have,

$$||f(x)| - |f(a)|| \leq |f(x) - f(a)|$$
$$\Rightarrow ||f(x)| - |f(a)|| < \epsilon, \text{ when } |x-a| < \delta$$

Hence $|f|$ is continuous.

(ii) The converse is NOT true. Consider the following example.

$$f(x) = \begin{cases} -1 & , x \leq 0 \\ +1 & , x > 0 \end{cases}$$

The function f is discontinuous at 0. But $|f|(x) = |f(x)| = 1$, for all $x \in \mathbb{R}$ is constant function and so $|f|$ is continuous.

(9) Suppose f and g are two continuous functions.

(i) To show that $\max(f, g)$ is continuous. Suppose $x_0 \in \mathbb{R}$ is arbitrary. To show that $\max(f, g)$ is continuous at x_0.

Case 1. $f(x_0) \neq g(x_0)$

Since f and g are continuous at x_0 for given $\epsilon = \frac{|f(x_0)-g(x_0)|}{2} > 0$ there exist $\delta_1 > 0$ and δ_2 such that,

$$|f(x) - f(x_0)| < \epsilon, \text{ if } |x - x_0| < \delta_1 \text{ and}$$
$$|g(x) - g(x_0)| < \epsilon, \text{ if } |x - x_0| < \delta_2.$$

Let us define $\delta = \min\{\delta_1, \delta_2\}$. Then $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$, $g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$, if $x \in (x_0 - \delta, x_0 + \delta)$. Without loss of generality assume that $f(x_0) > g(x_0)$.Then,

$$f(x_0) - \epsilon < \max\{f(x), g(x)\} < f(x_0) + \epsilon, \text{ if } x \in (x_0 - \delta, x_0 + \delta).$$
$$\Rightarrow |\max\{f(x), g(x)\} - \max\{f(x_0), g(x_0)\}| < \epsilon, \text{ if } |x - x_0| < \delta.$$

Hence $\max\{f(x), g(x)\}$ is continuous at x_0.

Case 2. $f(x_0) = g(x_0) = max\{f(x_0), g(x_0)\}$.

Since f and g are continuous at x_0 for given $\epsilon > 0$ there exist $\delta_1 > 0$ and δ_2 such that,

$$|f(x) - f(x_0)| < \epsilon, \text{ if } |x - x_0| < \delta_1 \text{ and}$$
$$|g(x) - g(x_0)| < \epsilon, \text{ if } |x - x_0| < \delta_2.$$

Let us define $\delta = \min\{\delta_1, \delta_2\}$. Then $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$, $g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$, if $x \in (x_0 - \delta, x_0 + \delta) \Rightarrow |\max\{f(x), g(x)\} -$

$\max\{f(x_0), g(x_0)\}| < \epsilon$, if $|x - x_0| < \delta$ and hence $\max\{f, g\}$ is continuous.
(ii) If a function F is continuous then $-F$ is continuous (check it!).
Since the functions f, g are continuous.
$\Rightarrow -f$, $-g$ are continuous.
$\Rightarrow \max\{-f, -g\}$ is continuous.
$\Rightarrow -\max\{-f, -g\}$ is continuous.
Now notice that $\min\{f, g\} = -\max\{-f, -g\}$ is continuous, which is what we want to prove.

(10) Let us define

$$g(x) := \max\{f(x), 0\}$$

and

$$h(x) := -\min\{f(x), 0\}.$$

Then $f = g - h$ (check it!) and the functions g and h are continuous follows from the previous problem. Also the functions g, h are non-negative.