Solutions to Exercises 3

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1. **Extra problem:** If $|a| < 1$ then $\lim_{n\to\infty} x_n = 0$ where $x_n = a^n$.
Proof: If $a = 0$, then there is nothing to prove. So let $0 < a < 1$ then $x_{n+1} = a \cdot x_n < x_n$. So, the sequence $\{x_n\}$ is monotonically decreasing sequence and bounded by 0. Hence the sequence is convergent and let l be the limit. So $\lim_{n\to\infty} x_{n+1} = a \cdot \lim_{n\to\infty} x_n$. Hence $l = a \cdot l$. Since $a \neq 0 \Rightarrow l = 0$. If $-1 < a < 0$, then take $c = -a$. By previous paragraph, $\lim_{n\to\infty} c^n = 0 \Leftrightarrow \lim_{n\to\infty}(-a)^n = 0$. Hence $\lim_{n\to\infty} a^n = 0$.

Now I start with assignment problems.

(a) $|a_n| = |\frac{n^2-(n+1)^2}{n(n+1)}| = \frac{2n+1}{n(n+1)} \leq \frac{2n+n}{n(n+0)} = \frac{3}{n}$
Now for every $\epsilon > 0$ we can find a natural number N (take any natural number greater than $\frac{3}{\epsilon}$) such that $|a_n - 0| < \epsilon \ \ \forall\, n > N$. So $\lim_{n\to\infty} a_n = 0$.
Alternative solution, $a_n = \frac{n^2-(n+1)^2}{n(n+1)} = \frac{-2n-1}{n(n+1)}$. Let $x_n = -(\frac{2}{n} + \frac{1}{n^2})$ and $y_n = (1 + \frac{1}{n})$. Then show that, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 1$ (use 1.6.a (means problem 6(a) in Ex.1), 1.6.c, 1.6.d). So $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(x_n/y_n) = (\lim_{n\to\infty} x_n)/(\lim_{n\to\infty} y_n) = \frac{0}{1} = 0$.

(b) See $a_n = 1$ for $n = 4m$ and $a_n = -1$ for $n = 4m-2$ where $m \in \mathbb{N}$. So for any real number l, $\max\{|a_{4m} - l|, |a_{4k-2} - l|\} \geq 1\ \forall\, m, k \in \mathbb{N}$. Now if we choose $0 < \epsilon < \frac{1}{2}$, we can not find a natural number N such that $|a_n - l| < \epsilon = \frac{1}{2} \ \ \forall\, n > N$. So $\lim_{n\to\infty} a_n$ does not exist.

(c) See $n^2 < 3^n$ (though it looks obvious, you can try to prove it). So $|a_n| = \frac{n}{3^n} < \frac{n}{n^2} = \frac{1}{n}$.
Now for every $\epsilon > 0$ we can find a natural number N (take any natural number greater than $\frac{1}{\epsilon}$) such that $|a_n - 0| < \epsilon \ \ \forall\, n > N$. So $\lim_{n\to\infty} a_n = 0$.

(d) $|a_n| = |\frac{(-1)^n+(1+(-1)^n)}{n}| \leq \frac{3}{n}$.
Now for every $\epsilon > 0$ we can find a natural number N (take any natural number greater than $\frac{3}{\epsilon}$) such that $|a_n - 0| < \epsilon \ \ \forall\, n > N$. So $\lim_{n\to\infty} a_n = 0$.

(e) See $|\sin(n^2)| \leq 1$. So $|a_n| = |\frac{n^{9/10}\sin(n^2)}{n+1}| < \frac{n^{9/10}}{n} = \frac{1}{n^{(1/10)}}$.
Now for every $\epsilon > 0$ we can find a natural number N (take any natural number greater than $\frac{1}{\epsilon^{10}}$) such that $|a_n - 0| < \epsilon \ \ \forall\, n > N$. So $\lim_{n\to\infty} a_n = 0$.

(f) Prove that, $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ (See 1.(i)). Since $f(x) = \log_a(x)$ is a continuous function on $(0,\infty)$, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$. If $x_n = n^{\frac{1}{n}}$, then $f(x_n) = a_n = \frac{\log_a n}{n}$.
So, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = \log_a(1) = 0$.

(g) See $a_n \leq \frac{1}{3}$ for $n = 4m$ and $a_n \geq \frac{9}{5}$ for $n = 4m-2$ where $m \in \mathbb{N}$. So for any real number l, $\max\{|a_{4m}-l|, |a_{4k-2}-l|\} \geq \frac{11}{15}\ \forall\, m,k \in \mathbb{N}$. Now if we choose $0 < \epsilon < \frac{1}{2}$, we can not find a natural number N such that $|a_n - l| < \epsilon = \frac{1}{2}\ \ \forall\, n > N$. So $\lim_{n\to\infty} a_n$ does not exist.

(h) (we assume a_n is a sequence of positive real numbers) If $a = 1$, then $a_n = 1$. Hence $\lim_{n\to\infty} a_n = 1$. If $a > 1$ then $a^{\frac{1}{n}} = 1 + h_n$ where $h_n > 0$. So $a = (1+h_n)^n > 1 + nh_n \Leftrightarrow 0 < h_n < \frac{a-1}{n} \Rightarrow 0 \leq \lim_{n\to\infty} h_n \leq \lim_{n\to\infty} \frac{a-1}{n} = 0$. Hence $\lim_{n\to\infty} h_n = 0 \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} a^{\frac{1}{n}} = 1 + \lim_{n\to\infty} h_n = 1$
If $a < 1$ then $\frac{1}{a} > 1$ and then $\lim_{n\to\infty}(\frac{1}{a})^{1/n} = 1 \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} a^{\frac{1}{n}} = 1$.

(i) Let $n^{\frac{1}{n}} = 1 + h_n$ where $h_n > 0$. So $n = (1+h_n)^n > 1 + \frac{n(n-1)}{2}{h_n}^2 \Leftrightarrow 0 < {h_n}^2 < \frac{2(n-1)}{n(n-1)} \Rightarrow 0 \leq \lim_{n\to\infty} {h_n}^2 \leq \lim_{n\to\infty} \frac{2}{n} = 0$. Hence $\lim_{n\to\infty} {h_n}^2 = 0 \Leftrightarrow \lim_{n\to\infty} h_n = 0 \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} a^{\frac{1}{n}} = 1 + \lim_{n\to\infty} h_n = 1$.

(j) Assume $p > q$, then $a_n = p(1 + (\frac{q}{p})^n)^{(1/n)}$. Now $1 < 1 + (\frac{q}{p})^n < 2\ \forall\, n \in \mathbb{N}$. $\Rightarrow p < p(1+(\frac{q}{p})^n)^{(1/n)} < p \cdot 2^{(1/n)}$.
$\Rightarrow p \leq \lim_{n\to\infty} p(1+(\frac{q}{p})^n)^{(1/n)} \leq p \cdot \lim_{n\to\infty} 2^{(1/n)} = p$ (since, $\lim_{n\to\infty} 2^{(1/n)} = 1$ (by 1.(h))). So $\lim_{n\to\infty} a_n = p$. If $q > p$ then $\lim_{n\to\infty} a_n = q$. If $q = p$ then $\lim_{n\to\infty} a_n = p \lim_{n\to\infty} 2^{\frac{1}{n}} = p$. Hence $\lim_{n\to\infty} a_n = \max\{p,q\}$.

(k) $a_n = n - \sqrt{(n+p)(n+q)} = \frac{(n-\sqrt{(n+p)(n+q)})(n+\sqrt{(n+p)(n+q)})}{n+\sqrt{(n+p)(n+q)}} = -\frac{(p+q)n+pq}{n+\sqrt{(n+p)(n+q)}}$. So, $a_n = \frac{-(p+q)-(pq/n)}{1+\sqrt{(1+p/n)(1+q/n)}}$. Now let $x_n = -(p+q)-(pq/n)$ and $y_n = 1+\sqrt{(1+p/n)(1+q/n)}$. Then show that $\lim_{n\to\infty} x_n = -(p+q)$ and $\lim_{n\to\infty} y_n = 2$ using 1.6.a and 1.6.c. Then $\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} = -(p+q)/2$ using 1.6.d.

(l) $a_n = \frac{3^n(1+(-2/3)^n)}{3^{n+1}(1+(-2/3)^{n+1})} = \frac{(1+(-2/3)^n)}{3(1+(-2/3)^{n+1})}$. Now let $x_n = 1 + (-2/3)^n$ and $y_n = 3(1+(-2/3)^{n+1})$.Then show that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} y_n = 3$ using 1.6.a and 1.6.c. Then $\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} = \frac{1}{3}$ using 1.6.d.

2. Let $x_n = \frac{2}{n^2}$ and $y_n = 1+\frac{1}{n^3}$, then $\lim_{n\to\infty} y_n = 1$ and $\lim_{n\to\infty} x_n = 0$. So by 1.6.$a$, $\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} = \frac{0}{1} = 0$.

Alternative solution, for any $\epsilon > 0$, $|a_n - 0| = \frac{2n}{n^3+1} < \frac{2n}{n^3} = \frac{2}{n^2} < \epsilon \ \forall\ n > N$ where N is any natural number greater than $\sqrt{\frac{2}{\epsilon}}$. So $L = 0$. Now we can choose $N = 5 > \sqrt{\frac{2}{0.1}}$ in the first case and $N = 150 = \sqrt{22500} > \sqrt{\frac{2}{0.0001}}$ in the second case.
See $a_n = (-9/10)^n$ converges to zero (see the extra problem). Now Now for every $\epsilon > 0$ we can find a natural number N (take any natural number greater than $\frac{\log \epsilon}{\log(9/10)}$) such that $|a_n - 0| < \epsilon \ \ \forall\ n > N$. Now putting the value of ϵ we can find the value of N. You can check $N = 22$ when $\epsilon = 0.1$ and $N = 4 \times 22 = 88$ when $\epsilon = 0.0001$
Comment: This problem shows how N depends on ϵ.

3. Given that $\lim_{n\to\infty} a_n = 0$. So for any $\epsilon > 0$ there exists a natural number N such that $|a_n - 0| < \sqrt{\epsilon} \quad \forall\ n > N$
$\Leftrightarrow |a_n| < \sqrt{\epsilon} \quad \forall\ n > N$
$\Leftrightarrow |a_n|^2 < \epsilon \quad \forall\ n > N$
$\Leftrightarrow |a_n^2 - 0| < \epsilon \quad \forall\ n > N$. So $\lim_{n\to\infty} a_n^2 = 0$
Comment: This solution shows that converse of the statement is also true.

4. (a) See $a_n = \frac{n}{(4n-3)(4n-1)} > \frac{n}{4n\cdot 4n} = \frac{1}{16n}$. Now $\Sigma a_n > \frac{1}{16}\Sigma\frac{1}{n}$. Since $\Sigma\frac{1}{n}$ is divergent, Σa_n is divergent.
   (b) See if $f(x) = \log x - x^{1/3}$ then $f'(x) = \frac{(3-x^{1/3})}{3x} < 0\ \forall x > 27$. So $f(x)$ is an decreasing function in $(27,\infty)$. So, $f(n) < f(28)$ $(= K)\ \forall n > 28$. Hence $\log n < n^{1/3} + K\ \forall n > 28$.
   Now $|a_n| < \frac{\sqrt{2n-1}(k+\sqrt[3]{4n+1})}{n(n+1)} (= c_n)\ \forall n > 28$. Take $b_n = \frac{1}{n^{(7/6)}}$, then $\lim_{n\to\infty} \frac{c_n}{b_n} = \sqrt{2}\sqrt[3]{4} (\neq 0, \infty)$. Since Σb_n is convergent, Σc_n is convergent. Hence Σa_n is convergent.
   (c) See $a_n = \frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$. Now $\Sigma a_n \leq \Sigma\frac{1}{n^2}$. Since $\Sigma\frac{1}{n^2}$ is convergent, Σa_n is convergent.
   (d) Here $\frac{a_{n+1}}{a_n} = \frac{n+2}{2^{n+1}}\frac{2^n}{n+1} = \frac{1+2/n}{2\cdot(1+1/n)}$. Now $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1+2/n}{2\cdot(1+1/n)} = \frac{1}{2} < 1$, hence Σa_n is convergent.
   (e) Take $b_n = \frac{1}{n^2}$, then $\frac{a_n}{b_n} = \frac{n^2}{(n+2)(n+1)} = \frac{1}{(1+2/n)(1+1/n)}$.
   $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{(1+2/n)(1+1/n)} = 1$.
   Since Σb_n is convergent, Σa_n is convergent.
   (f) See if $f(x) = \log x - x^{1/3}$ then $f'(x) = \frac{(3-x^{1/3})}{3x} < 0\ \forall x > 27$. So $f(x)$ is an decreasing function in $(27,\infty)$. So, $f(n) < f(28)$ $(= K)\ \forall n > 28$. Hence $\log n < n^{1/3} + K\ \forall n > 28$.
   Now $|a_n| = \frac{\log n}{n\sqrt{n+1}} < \frac{n^{1/3}+K}{n\sqrt{n+1}} (= c_n)\ \forall n > 28$. Take $b_n = \frac{1}{n^{(7/6)}}$, then $\lim_{n\to\infty}\frac{c_n}{b_n} = 1 (\neq 0, \infty)$. Since Σb_n is convergent, Σc_n is convergent. Hence Σa_n is convergent.

(g) See $\lim_{n\to\infty} a_n = 0$ iff $s > 1$. Since $\lim_{n\to\infty} a_n = 0$ is the necessary condition for convergence of the series Σa_n, we must have $s > 1$. Now if $f(x) = \log x - x^{1/s}$ then $f^{'}(x) = \frac{(s-x^{1/s})}{sx} < 0 \ \forall x > N$(some natural number). So $f(x)$ is an decreasing function in (N, ∞). So, $f(n) < f(N+1) \ (= K) \ \forall n > N+1$. Hence $\log n < n^{1/s}+K \ \forall n > N+1$ i.e. $(\log n)^s < (n^{1/s}+K)^s \ \forall n > N+1$ Now $|a_n| = \frac{1}{(\log n)^s} > \frac{1}{(n^{1/s}+K)^s} (= c_n) \ \forall n > N+1$. Take Take $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{c_n}{b_n} = 1 (\neq 0, \infty)$. Since Σb_n is divergent, Σc_n is divergent. Hence Σa_n is divergent.

(h) If $f(x)$ is a monotone decreasing function, we know that $f(x) \leq f(n)$ for $x \in [n, \infty]$ and $f(n) \leq f(x)$ for $x \in [N, n]$, hence for every large $n > N$,
$\int_n^{n+1} f(x)dx \leq \int_n^{n+1} f(n)dx = f(n) = \int_{n-1}^n f(n)dx \leq \int_{n-1}^n f(x)dx$
Since the lower estimate is also valid for $f(N)$, we get by summation over all n from N to some larger integer M.
$\int_N^{M+1} f(x)dx \leq \Sigma_N^M f(n) \leq f(N) + \int_N^M f(x)dx$
$\int_{N+1}^{\infty} f(x)dx < \Sigma_N^{\infty} f(n) \leq f(N) + \int_N^{\infty} f(x)dx$
See if $s = 0$ then $\Sigma|a_n| = \Sigma_2^{\infty} \frac{1}{\{n \log n\}} > \int_4^{\infty} \frac{dx}{\{x \log x\}} = \lim_{b\to\infty} \int_4^b \frac{dx}{\{x \log x\}} = \lim_{b\to\infty}\{\log\log b - \log\log 4\} \to +\infty$.
If $s < 0$ then $\Sigma|a_n| = \Sigma_2^{\infty} \frac{1}{\{n \log n(\log\log n)^s\}} > \Sigma_2^{\infty} \frac{1}{\{n \log n\}} > \int_4^{\infty} \frac{dx}{\{x \log x\}} = \lim_{b\to\infty} \int_4^b \frac{dx}{\{x \log x\}} = \lim_{b\to\infty}\{\log\log b - \log\log 4\} \to +\infty$
If $0 < s < 1$ then $\Sigma|a_n| = \Sigma_2^{\infty} \frac{1}{\{n \log n(\log\log n)^s\}} > \int_4^{\infty} \frac{dx}{\{x \log x(\log\log x)^s\}} = \lim_{b\to\infty} \int_4^b \frac{dx}{\{x \log x(\log\log x)^s\}} = \lim_{b\to\infty} \frac{1}{1-s}\{(\log\log b)^{1-s} - (\log\log 4)^{1-s} \to +\infty$.

If $s = 1$ then $\Sigma|a_n| = \Sigma_2^{\infty} \frac{1}{\{n \log n(\log\log n)\}} > \int_4^{\infty} \frac{dx}{\{x \log x(\log\log x)\}} = \lim_{b\to\infty} \int_4^b \frac{dx}{\{x \log x(\log\log x)\}}$
$= \lim_{b\to\infty}\{\log\log\log b - \log\log\log 4\} \to +\infty$.
If $s > 1$ then $\Sigma_2^{\infty}|a_n| = \Sigma_2^{\infty} \frac{1}{\{n \log n(\log\log n)^s\}} \leq f(2) + \int_2^{\infty} \frac{dx}{\{x \log x(\log\log x)^s\}}$. Now, $\int_2^{\infty} \frac{dx}{\{x \log x(\log\log x)^s\}} = \lim_{b\to\infty} \int_2^b \frac{dx}{\{x \log x(\log\log x)^s\}}$
$= \frac{1}{1-s} \lim_{b\to\infty}\{(\log\log b)^{1-s} - (\log\log 2)^{1-s}\} \to +\frac{(\log\log 2)^{1-s}}{s-1}$.
Hence the series is convergent only for $s > 1$.
Comment: Here a_1 does not exist, since $\log x$ is defined on $(0, \infty)$ and $\log 1 = 0$ ($\log\log n$ is not valid for n=1).

(i) See $a_n = \frac{1}{500n+4} > \frac{1}{500n+5n} = \frac{1}{505}\frac{1}{n}$. Now $\Sigma a_n > \frac{1}{505}\Sigma\frac{1}{n}$. Since $\Sigma\frac{1}{n}$ is divergent, Σa_n is divergent.

(j) Here $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{(1+1/n)^2}{(2+2/n)(2+1/n)}$. Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(1+1/n)^2}{(2+2/n)(2+1/n)} = \frac{1}{4} < 1$, hence Σa_n is convergent.

(k) See $a_n = \frac{1}{e^{n^2}} < \frac{1}{n^2}$. Now $\Sigma a_n < \Sigma \frac{1}{n^2}$. Since $\Sigma \frac{1}{n^2}$ is convergent, Σa_n is convergent.

(l) Here $\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{n^n}{3^n n!} = \frac{3}{(1+1/n)^n}$. Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{(1+1/n)^n} = \frac{3}{e} > 1$, hence Σa_n is divergent.

(m) See $a_n = \frac{n^{n+1/n}}{(n+1/n)^n} = \frac{n^{1/n}}{(1+(1/n)^2)^n} = \frac{n^{1/n}}{\sqrt{(1+(1/n)^2)^{n^2}}}$

Now $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^{1/n}}{\sqrt{(1+(1/n)^2)^{n^2}}} = \frac{1}{\sqrt{e}} \neq 0$, so Σa_n is divergent.

5. Since $\{x_n\}$ be a sequence of positive real numbers, $l \geq 0$. Hence $0 \leq l < 1$ and there exists a real number r such that $0 \leq l < r < 1$. Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$, for every $\epsilon > 0$ there exists natural number N_ϵ such that $|\frac{x_{n+1}}{x_n} - l| < \epsilon \quad \forall\ n > N_\epsilon$. Take $\epsilon = r - l$ then there exists natural number N such that $|\frac{x_{n+1}}{x_n} - l| < r - l \quad \forall\ n > N$. So, $|\frac{x_{n+1}}{x_n}| \leq |\frac{x_{n+1}}{x_n} - l| + |l| < r - l + l = r \quad \forall\ n > N$. So, $|\frac{x_{N+2}}{x_{N+1}}| < r,\ |\frac{x_{N+3}}{x_{N+2}}| < r,\ |\frac{x_{N+4}}{x_{N+3}}| < r, \cdots, |\frac{x_{N+p+1}}{x_{N+p}}| < r\ \forall\ p \in \mathbb{N}$
Multipling all of those we have, $|\frac{x_{N+p+1}}{x_{N+1}}| < r^p\ \forall\ p \in \mathbb{N}$
So, $0 \leq |x_{N+p+1}| < |x_{N+1}| r^p \Rightarrow 0 \leq \lim_{p\to\infty} |x_{N+p+1}| \leq |x_{N+1}| \lim_{p\to\infty} r^p \Rightarrow 0 \leq \lim_{p\to\infty} |x_{N+p+1}| \leq |x_{N+1}| \cdot 0$. Hence, $\lim_{n\to\infty} |x_n| = 0 \Leftrightarrow \lim_{n\to\infty} x_n = 0$.
Comment: This prove shows that if Σx_n is convergent then $\lim_{n\to\infty} x_n = 0$, which I use to prove $4(g)$.

6. Show that $|x_n| \leq 2$ (you can use mathematical induction if you know it). Now $x_{n+1}^2 - x_n^2 = 2 + x_n - 2 - x_{n-1} = x_n - x_{n-1}$. Since $x_2 > x_1 \Rightarrow x_3^2 - x_2^2 \Leftrightarrow x_3 > x_2$ and so on $\cdots$. So $\{x_n\}$ is a monotonically increasing sequence. Since it is bounded above, it is convergent. Let it converses to l. Then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n}$. So $l = \sqrt{2 + l} \Leftrightarrow l^2 = l + 2 \Leftrightarrow (l-2)(l+1) = 0 \Leftrightarrow\ l = 2$ or $l = -1$. But $\{x_n\}$ is a sequence of positive numbers, limit can not ne negative $\Rightarrow l = 2$.

7. If $u \in A$ then take $x_n = u$. So let $u \notin A$, then by definition of sup$A$, there exists $x_1 \in A$ such that $u - 1 < x_1 < u$. If $u - \frac{1}{2} < x_1 < u$, then take $x_2 = x_1$ otherwise there exists $x_2 (> x_1) \in A$ such that $u - \frac{1}{2} < x_2 < u$. Proceeding similar manner we get a sequence $\{x_n\}$ such that $x_n \leq x_{n+1}\ \forall\ n$ and $u - \frac{1}{n} < x_n < u < u + \frac{1}{n} \Rightarrow |x_n - u| < \frac{1}{n}$. Now for every $\epsilon > 0$ we can find a natural number N (take any natural number greater than $\frac{1}{\epsilon}$) such that $|a_n - u| < \epsilon \ \ \forall\ n > N$. So $\lim_{n\to\infty} a_n = u$.

8. This sequence is monotonically increasing. So it is enough to prove that the sequence is bounded above.

Now $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$
$\leq 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n-1\cdot n}$
$= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) = 2 - \frac{1}{n} < 2.$

9. For each natural number k there exists $x_{n_k} \in \{x_n\}$ such that $|x_{n_k}| > k$ i.e. $|\frac{1}{x_{n_k}}| < \frac{1}{k}$. Now for every $\epsilon > 0$, we can find a natural number N (Take any natural number greater than $\frac{1}{\epsilon}$) such that $|\frac{1}{x_{n_k}} - 0| < \epsilon \ \ \forall\ k > N$. Hence, $\lim_{k\to\infty} \frac{1}{x_{n_k}} = 0$.

10. $|x_n - x_{n-1}| = \frac{1}{2}|x_{n-1} - x_{n-2}|$.
Now, $|x_{n+p} - x_n| \leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \cdots + |x_{n+1} - x_n|$
$= \frac{1}{2^{n+p-2}}|x_2 - x_1| + \frac{1}{2^{n+p-3}}|x_2 - x_1| + \cdots + \frac{1}{2^{n-1}}|x_2 - x_1|$
$= \frac{|x_2 - x_1|}{2^{n-1}}[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{p-1}}]$
$< \frac{|x_2 - x_1|}{2^{n-1}}[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots]$
$= \frac{|x_2 - x_1|}{2^{n-1}} \cdot 2 = \frac{|x_2 - x_1|}{2^{n-2}} < \epsilon \ \ \forall\ p \geq 1$ and $n > N$ where N be any natural number greater than $(\log \frac{|x_2 - x_1|}{\epsilon})/(\log 2) + 2$.
Hence $\{\mathbf{x}_n\}$ be a cauchy sequence in $\mathbb{R}$, so the sequence is convergent.
See $x_3 = \frac{x_1}{2} + \frac{x_2}{2}$; $x_4 = \frac{x_1}{2^2} + \frac{x_2}{2} + \frac{x_2}{2^2}$
$x_5 = \frac{x_1}{2^2} + \frac{x_1}{2^3} + \frac{x_2}{2} + \frac{x_2}{2^3}$; $x_6 = \frac{x_1}{2^2} + \frac{x_1}{2^4} + \frac{x_2}{2} + \frac{x_2}{2^3} + \frac{x_2}{2^4}$
$x_7 = \frac{x_1}{2^2} + \frac{x_1}{2^4} + \frac{x_1}{2^5} + \frac{x_2}{2} + \frac{x_2}{2^3} + \frac{x_2}{2^5}$; $x_8 = \frac{x_1}{2^2} + \frac{x_1}{2^4} + \frac{x_1}{2^6} + \frac{x_2}{2} + \frac{x_2}{2^3} + \frac{x_2}{2^5} + \frac{x_2}{2^6}$
$x_9 = \frac{x_1}{2^2} + \frac{x_1}{2^4} + \frac{x_1}{2^6} + \frac{x_1}{2^7} + \frac{x_2}{2} + \frac{x_2}{2^3} + \frac{x_2}{2^5} + \frac{x_2}{2^7}; \cdots$
So, $x_{2n+1} = (\frac{x_1}{2^2} + \frac{x_1}{2^4} + \frac{x_1}{2^6} + \cdots + \frac{x_1}{2^{2n-2}}) + \frac{x_1}{2^{2n-1}} + (\frac{x_2}{2} + \frac{x_2}{2^3} + \frac{x_2}{2^5} + \cdots + \frac{x_2}{2^{2n-1}})$
Since $\{\mathbf{x}_n\}$ is a convergent sequence, it's subsequence $\{\mathbf{x}_{2n+1}\}$ is also convergent and converges to the same limit. So, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n+1} = (\frac{x_1}{2^2} + \frac{x_1}{2^4} + \frac{x_1}{2^6} + \cdots) + 0 + (\frac{x_2}{2} + \frac{x_2}{2^3} + \frac{x_2}{2^5} + \cdots)$
$= \frac{x_1}{4}\frac{1}{1-(1/4)} + \frac{x_2}{2}\frac{1}{1-(1/4)} = \frac{x_1 + 2x_2}{3}$.
Comment: You can also try to find the limit from the subsequence $\{x_{2n}\}$.