Solutions to Exercises 14

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1. Let $(x, y), (u, v) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.
(a) $T(\lambda(x, y) + (u, v)) = T(\lambda x + u, \lambda y + v) = (\lambda y + v, \lambda x + u) = \lambda(y, x) + (v, u) = \lambda T(x, y) + T(u, v)$. Hence T is linear.
(b) T is not linear. For example $T(2, 2) = (4, 4) \neq T(1, 1) + T(1, 1) = (2, 2)$.
(c) Suppose T maps each point on to its reflection with respect to a fixed line L through the origin. If L is given by $L := y = mx$ then T is given by the following,

$$T(x, y) = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now it is straight forward to show that, $T(\lambda(x, y)+(u, v)) = \lambda T(x, y)+T(u, v)$ and hence T is linear.

2. Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$
(a) T is linear as $T(\lambda(x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda x_3 + y_3) = (\lambda x_3 + y_3, \lambda x_2 + y_2, \lambda x_1 + y_1) = \lambda(x_3, x_2, x_1) + (y_3, y_2, y_1) = \lambda T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$.
$N(T) = \{0\}$ and $R(T) = \mathbb{R}^3$ and so the nullity of T is 0 and rank of T is 3.
(b) T is not linear: eg. $T(0, 2, 0) = (0, 4, 0), T(0, 1, 0) = (0, 1, 0) \Rightarrow T(0, 2, 0) \neq T(0, 1, 0) + T(0, 1, 0)$.
(c)
$$T(x, y, z) = (x + z, 0, x + y) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
It is easy to check that $T(\lambda(x_1, x_2, x_3)+(y_1, y_2, y_3)) = \lambda T(x_1, x_2, x_3)+T(y_1, y_2, y_3)$ and so T is linear.
$N(T) = \{(x, y, z) \in \mathbb{R}^3 | x + z = 0, x + y = 0\}$
$= \{(x, -x, -x) \in \mathbb{R}^3\} = \{x(1, -1, -1) | x \in \mathbb{R}\} = LS\{(1, -1, -1)\}$.
$R(T) = \{(x + z, 0, x + y) \in \mathbb{R}^3\} = \{(a, 0, b) | a, b \in \mathbb{R}\}$.
$Nullity(T) = 1$, $Rank(T) = 2$.

3. $S, T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ linear transformations given by, $T(x, y, z) = (x, x + y, x + y + z), S(x, y, z) = (z, y, x)$. Then $S^{-1} = S$ and $T^{-1}(x, y, z) = (x, y - x, z - y)$.
(a) $ST(x, y, z) = S(x, x + y, x + y + z) = (x + y + z, x + y, x)$
$TS(x, y, z) = T(z, y, x) = (z, z + y, z + y + x)$

$(ST-TS)(x,y,z) = ST(x,y,z) - TS(x,y,z) = (x+y, x-z, -y-z)$.
(b)
$S(x,y,z) = 0 \Rightarrow (z,y,x) = 0 \Rightarrow x = y = z = 0$,hence S is one-to-one.
$S^{-1} = S$, $T^{-1}(x,y,z) = (x, y-x, z-y)$, $(ST)^{-1} = T^{-1}S^{-1}(u,v,w) = T^{-1}(w,v,u) = (w, v-w, u-v)$, $(TS^{-1}(u,v,w) = S^{-1}T^{-1}(u,v,w) = S^{-1}(u, v-u, w-v) = (w-v, v-u, u)$
(c)
For $n = 1$, $(T-I)(x,y,z) = (x, x+y, x+y+z) - (x,y,z) = (0, x, x+y)$
For $n = 2$, $(T-I)^2(x,y,z) = (T-I)(0, x, x+y) = (0,0,x)$
For $n = 3$, $(T-I)^3(x,y,z) = (0,0,0)$ and hence $(T-I)^n = 0$, for all $n > 2$.

4. Suppose S,T $:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ are linear transformations such that $ST - TS = I$.
Now $ST^n - T^nS = (TS+I)T^{n-1} - T^nS = TST^{n-1} + T^{n-1} - T^nS$ $= T(TS+I)T^{n-2} + T^{n-1} - T^nS = T^2ST^{n-2} + 2T^{n-1} - T^nS$ $=.....= T^nS + nT^{n-1} - T^nS = nT^{n-1}$

5. The matrix $m(T)$ for the projection T is given by,

$$m(T) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that

$$T(x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

6. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear, so $T(e_1+e_2) = T(e_1)+T(e_2)$ and $T(3e_1+2e_2) = 3T(e_1) + 2T(e_2)$. Hence we have following two equations,
$T(e_1) + T(e_2) = 3e_1 + 9e_2$ and
$3T(e_1) + 2T(e_2) = 7e_1 + 23e_2$.
Solving above two equations we have $T(e_1) = e_1 + 5e_2, T(e_2) = 2e_1 + 4e_2$.
(a) $T(e_2 - e_1) = T(e_2) - T(e_1) = e_1 - e_2$
(b) The matrix of T with respect to standard ordered basis of $\mathbb{R}^2$ is given by

$$m(T) = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$$

7. Let,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $A^2 = 0$, then $trace(A) = a + d = 0 \Rightarrow d = -a$, and A^2 is given by,

$$\begin{bmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{bmatrix}$$

Hence $a^2 + bc$ must be 0.

8.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, A is nonsingular $\Leftrightarrow det(A) = 0 \Leftrightarrow ad - bc = 0$. From the above equation it is follows that,

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

9. A and B are nonsingular does not imply that $A+B$ is not nonsingular. For example consider any nonsingular matrix A and $B = -A$, then B is also nonsingular. But $A + B = 0$ in singular matrix.

10. Given that $A^2 = A$. To show that, $(A + I)^k = I + (2^k - 1)A$. We will proceed by mathematical induction.
$step - 1.$ $k = 0$.
$(A+I)^0 = I$ and $I + (2^0 - 1)A = I$. So the statement is true for $k = 0$.
$step - 2.$ Assume that the statement is true for k.
$step - 3$ $(A + I)^{k+1} = (A + I)^k(A + I) = \{I + (2^k - 1)A\}(A + I) = A+I+(2^kA-A)(A+I) = A+I+2^kA^2+2^kA-A^2-A = I+(2^{k+1}-1)A$. (Using $A^2 = A$)