Solutions to Exercises 13

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1. $c_1(1+t,1-t)+c_2(1-t,1+t) = (0,0)$ gives $(1+t)c_1+(1-t)c_2 = 0$ and $(1-t)c_1+(1+t)c_2 = 0 \Rightarrow c_1+c_2 = 0$ and $t(c_1-c_2) = 0$. Hence if $t \neq 0$ then $c_1, c_2 = 0 \Leftrightarrow (1+t,1-t),(1-t,1+t)$ are linearly independent.

2. Any non empty singleton set is linearly independent. So $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$ are linearly independent subsets of $\{A,B,C,D\}$. Again $B = 2D \Rightarrow$ the set $\{B,D\}$ is linearly dependent. Also easy to see that $\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{B,C\}$, $\{C,D\}$ are linearly independent subset of $\{A,B,C,D\}$ since each pair has nontrivial vectors and no one is scalar multiple of other. Since all vectors are in $\mathbb{R}^2$ (of dimension 2), so linearly independent set contains at most 2 vectors. So these are all linearly independent subsets of $\{A,B,C,D\}$.

3. (a) is true and (b) is false

   (a) $c_1(A+B)+c_2(B+C)+c_3(A+C) = 0 \Rightarrow (c_1+c_3)A+(c_1+c_2)B+(c_2+c_3)C = 0 \Rightarrow (c_1+c_3) = (c_1+c_2) = (c_2+c_3) = 0$ since A,B,C are linerly independent. Hence $c_1 = c_2 = c_3 = 0 \Rightarrow (a)$ is true.

   (b) $c_1(A-B)+c_2(B+C)+c_3(A+C) = 0$ for $c_1 = 1,\ c_2 = 1$ and $c_3 = -1$. Hence $(A-B),(B+C),(A+C)$ are linearly dependent.

4. If a set S of three vectors is a basis of $\mathbb{R}^3$, then $L(S) = \mathbb{R}^3$. So e_1, e_2, e_3 are in $L(S)$. Conversly, if e_1, e_2, e_3 are in $L(S)$ then for all real numbers a, b, c the element $ae_1+be_2+ce_3$ is in $L(S)$, since $L(S)$ is a subspace of $\mathbb{R}^3$. Hence $(a,b,c) \in L(S)\ \forall a,b,c \in \mathbb{R}$. Hence $\mathbb{R}^3 \subset L(S)$. But $L(S)$ is a subspace of $\mathbb{R}^3$, so $L(S) = \mathbb{R}^3$. So S must be linearly independent set, since S has three vectors and dimension of $L(S) = 3$. So S is a basis of $\mathbb{R}^3$.

5. Check $\{(0,1,0),(0,1,1),(1,1,1)$ and $\{(0,0,1),(0,1,1),(1,1,1)$ are two bases of $\mathbb{R}^3$.

6. (a) Since $(2,1,0) = 2(1,1,1)-(0,1,2)$ and $(2,0,-2) = 2(1,0,-1)$, then $(2,1,0),(2,0,-2) \in L(S) \Rightarrow c_1(2,1,0)+c_2(2,0,-2) \in L(S)\ \forall\ c_1, c_2 \in \mathbb{R} \Rightarrow L(T) \subset L(S)$.

   (b) Since $(1,2,3) = (1,1,1)+(0,1,2)$ and $(1,3,5) = (1,1,1)+2(0,1,2)$, then $(1,2,3),(1,3,5) \in L(S) \Rightarrow c_1(1,2,3)+c_2(1,3,5) \in L(S)\ \forall\ c_1, c_2 \in \mathbb{R} \Rightarrow L(U) \subset L(S)$.
   Since $(1,2,3) = 2(2,1,0)-3/2(2,0,-2)$ and $(1,3,5) = 3(2,1,0)-5/2(2,0,-2)$ then $(1,2,3),(1,3,5) \in L(T) \Rightarrow c_1(1,2,3)+$

$c_2(1,3,5) \in L(T)\ \forall\ c_1, c_2 \in \mathbb{R} \Rightarrow L(U) \subset L(T)$. But U, T both are linearly independent set, hence both have same dimension. So $L(U) = L(T)$.

7. (a) Trivially, $B \subset L(B)$. Hence $A \subset B \subset L(B)$. So $\Sigma_{i=1}^{n} c_i a_i \in L(B)\ \forall\ c_i \in \mathbb{R},\ a_i \in A$, since $L(B)$ be a subspace of $\mathbb{R}^n \Rightarrow L(A) \subset L(B)$.
   (b) $A \cap B \subset A \Rightarrow L(A \cap B) \subset L(A)$ and $A \cap B \subset B \Rightarrow L(A \cap B) \subset L(B)$. So $L(A \cap B) \subset L(A) \cap L(B)$.
   (c) Take $A = \{e_1\}$ and $B = \{2e_1\}$. Then $A \cap B = \phi \Rightarrow L(A \cap B) = 0$ but $L(A) = L(B) = (x, 0, ..., 0), x \in \mathbb{R}$. Hence $L(A) \cap L(B) = (x, 0, ..., 0), x \in \mathbb{R}$.
8. Just check all properties of vector space.
9. Suppose dimension is finite say n. Let $\{x_1, x_2, ..., x_n\}$ be a basis for the vector space. Then any element of $\mathbb{R}$ can be expressed as $c_1x_1 + c_2x_2 + \cdots + c_nx_n \equiv (c_1, c_2, ..., c_n)$ where $c_1, c_2, ..., c_n \in \mathbb{Q}$. So $\mathbb{R} \equiv \mathbb{Q}^n$ which is a contradiction since $\mathbb{R}$ is a uncountable set but $\mathbb{Q}^n$ is a countable set. Hence dimension of the vector space must be infinite.
   Additional Exercise: Try to prove number of elements of a basis must be uncountable. It is a very good exercise.
10. Suppose dimension of V is n. Now choose any $n+1$ elements of S, say $\{x_1, x_2, ..., x_n, x_{n+1}\}$. If this set is linearly independent then V has a linearly independent set of cardinality $n+1$ which contradicts that dimension of V is n. So this set must be linearly dependent. Thus S has no linearly independent set of cardinality greater than n. Hence dimension of S must be less than or equal to n. Therefore S is finite dimensional and $dim(S) \leq dim(V)$.