Solutions to Exercises 12

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1. (a) $z + \bar{z} = 1$. If $z = x + iy$ then $z + \bar{z} = 2x = 1$ imply $x = 1/2$.

   (b) $|z-1| = |z+1|$ imply $|z-1|^2 = |z+1|^2$, substitute $z = x + iy$ in this to obtain $x = 0$ i.e. the y-axis.

   (c) $|z-i| = |z+i|$. Again substitute $z = x + iy$ and solve to obtain $y = 0$ i.e. the x-axis.

   (d) $z + \bar{z} = |z|^2$, i.e $2x = x^2 + y^2$ i.e. the unit circle at the point $(1, 0)$.

2. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ where a_i is real for all $i = 0, 1, ..., n$.

   (a) Then $\overline{f(z)} = a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + ... + a_0 = f(\bar{z})$ since $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$ and $\overline{z_1 z_2} = \bar{z_1}\bar{z_2}$.

   (b) So if $f(z_0) = 0$ for some $z_0 \in \mathbb{C}$ then $\overline{f(z_0)} = 0 = f(\bar{z_0})$, hence $\bar{z_0}$ is also a solution of f i.e. complex roots occur in conjugates.

3. Given $w = \frac{az+b}{cz+d}$ where a, b, c, d are real. Now $|cz+d|^2 = (cz+d)(c\bar{z}+d)$ i.e

$$w = \frac{ac|z|^2 + bc\bar{z} + adz + bd}{|cz+d|^2} \tag{1}$$

   Thus $w - \bar{w} = \frac{bc(\bar{z}-z)+ad(z-\bar{z})}{|cz+d|^2} = \frac{(ad-bc)(z-\bar{z})}{|cz+d|^2}$.
   Now if $z = x + iy$ from equation (1) imaginary part of w is equal to $\frac{(ad-bc)y}{|cz+d|^2}$. Hence imaginary part of z and w has the same sign if $ad - bc > 0$.

4. Note that $z_3 - z_2 = i(z_1 - z_2)$, since the angle between them at z_2 is $\pi/2$. Now square both sides and solve to get the required expression.

5. Equation of a straight line in $\mathbb{R}^2$ is $\alpha x + \beta y = c$. Define $b = \alpha/2 + i\beta/2$, and $z = x + iy$, then compute $b\bar{z} + \bar{b}z$ to obtain $\alpha x + \beta y$. Thus this straight line can be written as $b\bar{z} + \bar{b}z = c$. Similarly the other way.

6. (a) $X_4 = \alpha_1(1, 1, 1) + \alpha_2(0, 1, 1) + \alpha_3(1, 1, 0)$ i.e. $X_4 = (\alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2)$.

   (b) If $X_4 = (0, 0, 0)$ imply $\alpha_1 + \alpha_3 = 0$ i.e $-\alpha_1 = \alpha_3$, $\alpha_1 + \alpha_2 = 0$ i.e $-\alpha_1 = \alpha_2$. Now substituting the values in $\alpha_1 + \alpha_2 + \alpha_3 = -\alpha_1 = 0$. Thus each $\alpha_i = 0$ for every $i = 1, 2, 3$.

   (c) Solving the linear system of equation

$$\alpha_1 + \alpha_3 = 1$$
$$\alpha_1 + \alpha_2 + \alpha_3 = 2$$
$$\alpha_1 + \alpha_2 = 3$$

we get $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = -1$.

7. If $A = a_1 + ia_2$ and $C = c_1 + ic_2$ then by law of addition $B = (a_1 + c{+}_1) + i(a_2 + c_2)$ thus $\frac{1}{2}B = A + \frac{1}{2}(C - A)$. Note that $A + \frac{1}{2}(C - A)$ is the mid point of the diagonal joining A and C. So the geometric property of parallelogram following from this equation is

   The diagonals bisects each other.

8. If $X = (1, 0)$,$Y = (0, 1)$ and $Z = (0, 2)$ then $X.Y = X.Z = 0$ but $Y \neq Z$ and $X \neq 0$.. Hence the statement is not true when $n \geq 2$.

9. Suppose $X.Y = 0$ for every Y and $X \neq 0$ then if $X = (a_1, a_2, ..., a_n)$ then choose $Y = (a_1, a_2, ..., a_n)$ and $Y \neq 0$. Then $X.Y = a_1^2 + a_2^2 + ... + a_n^2 \neq 0$. Which is a contradiction!

10. Let $A = (a_1, a_2, ..., a_n), B = (b_1, b_2, ..., b_n)$, then
    $\|A + B\|^2 + \|A - B\|^2 = \Sigma_{i=1}^n (a_i + b_i)^2 + (a_i - b_i)^2 = \Sigma_{i=1}^n 2(a_i^2 + b_i^2) = 2(\|A\|^2 + \|B\|^2)$.
    In case of $\mathbb{R}^2$ note that $\|A + B\|$ and $\|A - B\|$ are the length of the diagonals of the parallelogram formed by the vectors A and B from origin. So this identity proves that in a parallelogram sum of squares of the length of the diagonals is twice the sum of squares of the length of the two sides.