Solutions to Exercises – 11
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1. (a)
$$f^{'}(x) = e^{(\int_0^x e^{-t^2}dt)}e^{-x^2}.$$
   (b)
$$f^{'}(x) = (\ln x)^{\ln x}(\frac{\ln\ln x}{x} + \frac{1}{x}).$$
   (c)
$$f^{'}(x) = x^x(1 + \ln x).$$

2. Put $y := \frac{1}{x}$. As $x \to \infty$ $y \to 0$. Thus
$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{y\to 0}\frac{\sin y}{y} = 1.$$

3. Since $f > 0$ on $[a, b]$, so $\log(f(a))$ and $\log(f(b))$ make sense. Hence $\int_a^b \frac{f^{'}(t)}{f(t)} = \log(f(t))|_a^b = \log(f(b)) - \log(f(a))$.

4. Using the integration by parts we have, $F(x) = \int_2^x \frac{1}{\ln t}dt = \frac{1}{\ln t}t|_2^x + \int_2^x \frac{1}{(\ln t)^2}\frac{1}{t}tdt = \frac{x}{\ln x} - \frac{2}{\ln 2} + \int_2^x \frac{1}{(\ln t)^2}dt \geq \frac{x}{\ln x} - \frac{2}{\ln 2}$.
   But we know that the limit of the function $\frac{x}{\ln x}$ diverges to infinity as x goes to infinity. Hence the function $F(x) = \int_2^x \frac{1}{\ln t}dt$ is unbounded.

5. Since $\int_0^\pi f(x)\sin(x)dx = -f(x)\cos(x)|_0^\pi + \int_0^\pi f^{'}(x)\cos(x)dx = -f(\pi)(-1) + f(0) + f^{'}(x)\sin(x)|_0^\pi - \int_0^\pi f^{''}(x)\sin(x)dx = 1 + f(0) + 0 - \int_0^\pi f^{''}(x)\sin(x)dx = 1 + f(0) - \int_0^\pi f^{''}(x)\sin(x)dx$.
   Given that $\int_0^\pi \left[f(x) + f^{''}(x)\right]\sin(x)dx = 0$. Hence from above we have $1 + f(0) = 0$. That is $f(0) = -1$.

6. Consider the integral $\int_a^b f(x)dx$. Put $t = a + b - x$. Then after putting this and changing the limits the integral becomes $\int_a^b f(x)dx = -\int_b^a f(t - a - b)dt$.
   Now using the fact that $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Then we have
$$\int_a^b f(x)dx = -\int_b^a f(t - a - b)dt = \int_a^b f(t - a - b)dt$$

7. First try to show that
$$f^{'}(0) = 1, f^{''}(0) = 1, f^{'''}(0) = 0.$$

We also know that the Taylor polynomial of function $f(x)$ of degree 3 at 0 is given by

$$f(0) + f^{'}(0)x + f^{''}(0)\frac{x^2}{2!} + f^{'''}(0)\frac{x^3}{3!}.$$

So we have

$$1 + x + \frac{x^2}{2!}.$$

8. First show that

$$\sin^{4n+1}(x) = \cos(x), \sin^{4n+2}(x) = -\sin(x), \sin^{4n+3}(x) = -\cos(x), \sin^{4n}(x) = \sin(x).$$

Then we have

$$\sin^{4n+1}(0) = 1, \sin^{4n+2}(0) = 0, \sin^{4n+3}(0) = -1, \sin^{4n}(0) = 0.$$

Now by using this we can write the Taylor polynomial of degree $2n+1$ for $g(x) = \sin(x)$ as

$$g(0) + g^{'}(0)x + g^{''}(0)\frac{x^2}{2!} + g^{'''}(0)\frac{x^3}{3!} + ... + g^{2n+1}(0)\frac{x^{2n+1}}{(2n+1)!}.$$

That is

$$x - \frac{x^3}{3!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

So to get the result for $f(x) = \sin(x^2)$ simply replace x by x^2 in the above, then we have

$$x^2 - \frac{(x^3)^2}{3!} + ... + (-1)^n \frac{(x^{2n+1})^2}{(2n+1)!}.$$

9. Use integration by parts and the fact that $e^x \geq 1 + x$ to show that the inequality hold for $n = 0$. Then use induction.

10. It is easy to see that

$$\int_0^x \frac{t^n}{1+t} dt = -\int_x^0 \frac{t^n}{1+t} dt.$$

So

$$|\int_0^x \frac{t^n}{1+t} dt| = |\int_x^0 \frac{t^n}{1+t} dt|.$$

As $-1 < x \leq t \leq 0$, so $1 \leq \frac{1}{1+t} \leq \frac{1}{1+x}$. Now using the fact that $|\int_x^0 f(t)dt| \leq |\int_x^0 |f(t)|dt|$, we have

$$|\int_x^0 \frac{t^n}{1+t} dt| \leq |\int_x^0 \frac{|t|^n}{|1+t|} dt| \leq |\int_x^0 \frac{(-t)^n}{|1+x|} dt|.$$

That is

$$|\int_x^0 \frac{(-t)^n}{|1+x|}dt| = |\frac{(-1)^n}{1+x}\int_x^0 (t)^n dt|.$$

But

$$\int_x^0 (t)^n dt = \frac{t^{n+1}}{1+n}|_x^0 = -\frac{x^{n+1}}{1+n}.$$

Hence by above we have

$$|\int_x^0 \frac{t^n}{1+t}dt| \leq |\frac{(-1)^n}{1+x}\int_x^0 (t)^n dt| = \frac{|x|^{n+1}}{(1+x)(1+n)}.$$