Solutions to Exercises 10

by Mizanur Rahaman

1. Note that $\int \frac{1}{\sqrt{1-t^2}} = \sin^{-1} t$. So putting the upper limits and lower limits we get $\sin^{-1}(\sin x) - \sin^{-1}(-\cos x) = x + \frac{\pi}{2} - x = \frac{\pi}{2}$.

2. First of all observe that by FTC $f'(x) = \cos\cos x$, and that this is never zero(why not?). So f' is either always positive or negative (why?). Hence f is one one, and hence has inverse. Now $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))}$. Note that $f^{-1}(0) = 1$, and so we have $\frac{1}{\cos\cos 1}$.

3.

4. Let us put $g(u) = \int_0^u f(t)dt$. Then the RHS becomes $\int_0^x g(u)du$. Now let us apply Integration by part observing that that g is diiferentiable and by FTC we have $g'(u) = f(u)$. Hence $\int_0^x g(u)du = \{ug(u) - \int ug'(u)du\}_0^x$. Now replace $g'(u) = f(u)$ and putting the limits we get the LHS.

5. $f(x) = \int_0^x \big(\int_0^y \big(\int_0^z \frac{1}{\sqrt{(1+\sin^2 t)}}dt\big)dz\big)dy$. By using FTC we have $f'(x) = \int_0^x \big(\int_0^z \frac{1}{\sqrt{(1+\sin^2 t)}}dt\big)dz$. Agian $f''(x) = \int_0^x \frac{1}{\sqrt{(1+\sin^2 t)}}dt$, and $f'''(x) = \frac{1}{\sqrt{(1+\sin^2 x)}}$.

6. $g(t) = \frac{1}{t} + \frac{1}{2t\sqrt{t}}(t > 0)$ this will give the required conclusion.

7. Let $x = n + \{x\}$, where $n = [x] \geq 0$. Then $\int_0^x [t]^2 dt = 2x - 2$ gives us $0^2 + 1^2 + \cdots (n-1)^2 + n^2\{x\} = 2n + 2\{x\} - 2$ which implies $\left[\frac{n(n-1)(2n-1)}{6} - 2n\right] + \{x\}(n^2 - 2) + 2 = 0$. Verify that for $n \geq 3$, above eqn has no soln. For $n = 2$ above eqn reduces to $\{x\} = \frac{1}{2}$. For $n = 1$ we have $\{x\} = 0$. For $n = 0$ there is no soln. Hence the soln is $x = 1, 2.5$.

8. When $x \geq 0\, t \in (0, x)$ and hence $|t| = t$ then the integrand is $4t^2$.So the integral is $\frac{4x^3}{3}$ which is equal to the RHS when $x \geq 0$. Similarly consider the case when $x \leq 0$.

9. Differentiate the equation both side and from FTC you will get the expression for $f(x) = 2x + sin(2x) + 2xsin(2x) + sec(2x)tan(2x)$. now differentiating again and putting the value $x = \frac{\pi}{4}$ we get the values.

10. Expanding we get $2f(x) = x^2 \int_0^x g(t)dt - 2x \int_0^x tg(t)dt + \int_0^x t^2 g(t)dt$. now by FTC we get $2f'(x) = x^2 g(x) + 2x \int_0^x g(t)dt - 2x^2 g(x) - 2\int_0^x tg(t)dt + x^2 g(x)$.this gives the form of $f'(x)$. also we get by FTC again that $f''(x) = \int_0^x g(t)dt$.so $f''(1) = 2$ by the given value.