Solutions to Exercises 1
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1. Let a and b be two rational numbers. Without loss of generality we assume that $a < b$. Take $a_1 := \frac{a+b}{2}$. Clearly a_1 is a rational number and lies between a and b (prove it!). Repeat this process to get the result.

2. Consider a countable union

$$S := \bigcup_{i=1}^{\infty} S_i$$

of countable sets $S_i, i = 1, 2, \ldots$. We have to show that the set S is countable. Since S_i is countable, we can write the elements of S_i as

$$a_{i1}, a_{i2}, a_{i3}, \ldots$$

Then we can arrange the elements of the union of the sets in a doubly-infinite array:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Now we can create a bijection by counting each element as we can follow the arrows:

$$\begin{array}{ccccccc} a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} \\ & \swarrow & & \nearrow & & \swarrow & \\ a_{21} & & a_{22} & & a_{23} & & \\ \downarrow & \nearrow & & \swarrow & & & \\ a_{31} & & a_{32} & & & & \\ & \swarrow & & & & & \\ a_{41} & & & & & & \end{array}$$

In other words, we have a pairing

$$\begin{array}{cccccccc} a_{11} & a_{12} & a_{21} & a_{31} & a_{22} & a_{13} & a_{14} & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{array}$$

of each element of S with a natural number, and so S is countably infinite.

Next we have to prove that $\mathbb{Q}$ is countable. Define for each $k \in \mathbb{N}$,

$$A_k := \left\{ \dots - \frac{2}{k}, -\frac{1}{k}, 0, \frac{1}{k}, \frac{2}{k}, \dots \right\}.$$

It is easy to see that

$$\mathbb{Q} = \bigcup_{k=1}^{\infty} A_k.$$

Then deduce that $\mathbb{Q}$ is countable.

3. Let S be an infinite set. Choose a_1 in S. Since S is infinite set there exists $a_2 \in S \setminus \{a_1\}$. Repeat this process to get the result.

4. Define

$$A := \{a \in \mathbb{Q} : a^2 < 2\}.$$

Then prove that $A \neq \emptyset$, sup $A = \sqrt{2}$ and inf $A = -\sqrt{2}$.

5. The set A is a singleton set. Let sup$(A)= a_0$ and inf $(A) = b_0$. Then by the definition of sup and inf of a set we have

$$b_0 \leq x \leq a_0$$

$\forall x \in A$. So if $a_0 = b_0$, then $x = a_0 = b_0$ $\forall x \in A$.

6. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

   (a) As the sequence $\{x_n\}$ converges to x, so for given $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$\mid x_n - x \mid < \frac{\epsilon}{2}$$

   for all $n \geq N_1$. Similarly for the sequence $\{y_n\}$ we have $N_2 \in \mathbb{N}$ such that

$$\mid y_n - y \mid < \frac{\epsilon}{2}$$

   for all $n \geq N_2$. Now define $N := \max(N_1, N_2)$ then we have

$$\mid x_n - x \mid < \epsilon \text{ and } \mid y_n - y \mid < \epsilon$$

   for all $n \geq N$.
   Now $\mid (x_n + y_n) - (x + y) \mid = \mid (x_n - x) + (y_n - y) \mid \leq \mid (x_n - x) \mid + \mid y_n - y \mid$.
   So by above for all $n \geq N$ we have

$$\mid (x_n + y_n) - (x + y) \mid < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

   So

$$\lim_{n\to\infty} (x_n + y_n) = x + y.$$

(b) Proof of this part using the proof of (e). Put $c = -1$ in (e) then we have

$$\lim_{n\to\infty} -y_n = -y.$$

Now apply (a) to the sequences $\{x_n\}$ and $\{-y_n\}$ then we get the result.

(c) Claim. If the sequence $\{x_n\}$ converges to x then the sequence $\{x_n^2\}$ converges to x^2.

Proof of the claim. We know that if the sequence $\{x_n\}$ converges to x then the sequence is bounded. That is there exists $M_1 > 0$ such that

$$| x_n | \leq M_1$$

for all $n \in \mathbb{N}$. Now define $M := M_1 + | x |$. Then

$$| x_n + x | \leq M$$

for all $n \in \mathbb{N}$. Now as the sequence $\{x_n\}$ converges to x, so for given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$| x_n - x | < \frac{\epsilon}{M}$$

for all $n \geq N$. So $| x_n^2 - x^2 | = | (x_n - x)(x_n + x) |$

$$\begin{aligned} \text{Note that } \ | x_n^2 - x^2 | &= | (x_n - x)(x_n + x) | \\ &= | x_n - x || x_n + x | \\ &\leq | x_n - x | M \quad (\text{since } \ | (x_n + x) | \leq M) \\ &< \frac{\epsilon}{M} M \qquad \text{for all } n \geq N(\text{ from above}) \\ &= \epsilon \end{aligned}$$

Thus for given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$| x_n^2 - x^2 | < \epsilon$$

for all $n \geq N$. Hence sequence $\{x_n^2\}$ converges to x^2.
Now $x_n y_n = \frac{1}{4}\left((x_n + y_n)^2 - (x_n - y_n)^2\right)$. From (a) we have $\lim_{n\to\infty}(x_n + y_n) = x + y$ and from (b) we have $\lim_{n\to\infty}(x_n - y_n) = x - y$. Now from above claim we have

$$\lim_{n\to\infty}(x_n + y_n)^2 = (x + y)^2 \text{ and } \lim_{n\to\infty}(x_n - y_n)^2 = (x - y)^2.$$

Hence

$$\lim_{n\to\infty} x_n y_n = \frac{1}{4}\left((x + y)^2 - (x - y)^2\right) = xy.$$

(d) Given that $y \neq 0$.
Claim. $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$.
Proof of the claim. As the sequence $\{y_n\}$ converges to y, so for given $\epsilon =\mid y \mid> 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\mid y_n - y \mid< \frac{\mid y \mid}{2}$$

for all $n \geq N_1$. Now $\mid y_n \mid=\mid y + y_n - y \mid \geq \mid y \mid - \mid y_n - y \mid > \frac{|y|}{2}$ for all $n \geq N_1$. That is

$$\mid \frac{1}{y_n} \mid< \frac{2}{\mid y \mid}$$

for all $n \geq N_1$. Next as the sequence $\{y_n\}$ converges to y, so for given $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\mid y_n - y \mid< \frac{\mid y \mid^2 \epsilon}{2}$$

for all $n \geq N_2$.
Now let $N :=$max(N_1, N_2) then for all $n \geq N$ we have

$$\begin{aligned}
\left|\frac{1}{y_n} - \frac{1}{y}\right| &= \left|\frac{y_n - y}{y_n y}\right| \\
&= \frac{|y_n - y \mid}{\mid y_n \mid\mid y \mid} \\
&\leq \frac{2 \mid y_n - y \mid}{\mid y \mid\mid y \mid} \quad \left(\text{since } \mid \frac{1}{y_n} \mid< \frac{2}{\mid y \mid}\right) \\
&< \frac{\mid y \mid^2 \epsilon}{2} \frac{2}{\mid y \mid^2} \quad \left(\text{since } \mid y_n - y \mid< \frac{\mid y \mid^2 \epsilon}{2}\right) \\
&= \epsilon
\end{aligned}$$

That is, for given $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| < \epsilon$$

for all $n \geq N$.
Hence

$$\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$$

Now define a sequence $\{z_n\}$ by

$$z_n := \frac{1}{y_n}.$$

By above claim we have $\lim_{n\to\infty} z_n = \frac{1}{y}$. Now by using part (c) we have

$$\lim_{n\to\infty} x_n z_n = \frac{x}{y}.$$

That is

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}.$$

(e) This follows directly from the definition.

7. Check that $N = 28$.

8. (a) Define $x_n := \frac{n^2}{n+5}$. It is easy to check that the sequence $\{x_n\}$ is a monotonically increasing positive sequence which is not bounded above. Hence the sequences diverges to ∞.

   (b) Define $x_n := \frac{3n}{n+7\sqrt{n}}$. That is $x_n = \frac{3}{1+\frac{7}{\sqrt{n}}}$.
   Now check that the sequence is non-decreasing and bounded above by 3. Hence it converges.

   (c) Define $x_n := \frac{3n}{n+7n^2}$. That is $x_n = \frac{3}{1+7n}$.
   Now note that the sequence is non-increasing and bounded below. Hence it converges.

9. Let $\{x_n\}$ be a convergent sequence with limits l and m. Now for given $\epsilon > 0$ there exist N_1 and N_2 in $\mathbb{N}$ such that $\mid x_n - l \mid < \frac{\epsilon}{2}$ for all $n \geq N_1$ and $\mid x_n - m \mid < \frac{\epsilon}{2}$ for all $n \geq N_2$.
   Define $N := \max(N_1, N_2)$. Then for all $n \geq N$ we have

$$\mid x_n - l \mid < \frac{\epsilon}{2} \text{ and } \mid x_n - m \mid < \frac{\epsilon}{2}.$$

   Now $|l - m| = |l - x_n + x_n - m|$

$$\begin{aligned} |l - m| &= |l - x_n + x_n - m| \\ &\leq |l - x_n| + |x_n - m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{(for all } n \geq N) \\ &= \epsilon \end{aligned}$$

   Thus $|l - m| < \epsilon$ and this true for any $\epsilon > 0$. That is $l = m$.

10. For a given $\epsilon > 0$, there exist N_1 and N_2 in $\mathbb{N}$ such that $\mid x_{2n} - l \mid < \epsilon$ for all $n > N_1$ and $\mid x_{2n-1} - l \mid < \epsilon$ for all $n > N_2$.
    Define $N := \max(N_1, N_2)$ then for all $n > N$, we have

$$\mid x_{2n} - l \mid < \epsilon \text{ and } \mid x_{2n-1} - l \mid < \epsilon.$$

    Now for any n greater then N it follows that

$$\mid x_n - l \mid < \epsilon.$$