UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 9

12 October, 2012

1. Consider the following step function on $[0, 2]$:

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 2, & 1 < x \leq 2, \end{cases}$$

For each $n \geq 2$, consider the partition of $[0, 2]$ given by $\mathcal{P}_n = \{t_0 = 0, t_1 = 1 - 1/n, t_2 = 1 + 1/n, t_3 = 2\}$. Compute the lower and upper sums $\mathcal{L}(\mathcal{P}_n)$, $\mathcal{U}(\mathcal{P}_n)$ corresponding to the partition $\mathcal{P}_n$. Show that for every $\varepsilon > 0$, it is possible to find n such that the difference between the upper and lower sums indicated above is no bigger than ε. Conclude that the given function f is integrable and calculate the value of

$$\int_0^2 f(x)\, dx.$$

2. For a positive integer n, find the value of $\int_0^n [x]\, dx$.

3. Evaluate without doing any computations:

$$\int_{-1}^{1} x^3 \sqrt{1 - x^2}\, dx.$$

4. Evaluate without doing any computations:

$$\int_{-1}^{1} (x^5 + 3)\sqrt{1 - x^2}\, dx.$$

5. Prove that

$$\int_0^x \frac{\sin t}{t + 1}\, dt > 0$$

for all $x > 0$.

6. If $a < b < c < d$ and f is integrable on $[a, d]$, prove that f is integrable on $[b, c]$.

7. Prove that if f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then $\int_a^b f(t)\, dt \geq \int_a^b g(t)\, dt$.

8. Prove that

$$\int_a^b f(x)\, dx = \int_{a+c}^{b+c} f(x - c)\, dx.$$

9. Find the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = \frac{x^2}{2} + 2$.

10. Find the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = 1 - x^2$.