UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 8

5 October, 2012

1. Obtain $r = \sqrt{15} - 3$ as an approximation to the non-zero root of the equation
$$x^2 = \sin x$$
by using the approximation $\sin x \approx P_{3,0}(x)$, where $P_{3,0}(x)$ is the degree three Taylor polynomial for $\sin x$ at $a = 0$.

2. Let f, g be functions on the real line for which Taylor's theorem can be applied. Fix a and let $P_{n,a}(x)$ and $Q_{n,a}(x)$ be the corresponding Taylor polynomials for f, g respectively. Show that the Taylor polynomials for $f + g$ at the point a are given by $P_{n,a}(x) + Q_{n,a}(x)$. Can you find the Taylor polynomials at a for the function $f(x)g(x)$?

3. Find f^{-1} for each of the following f.

   (a)
$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   (b) $f(x) = x + [x]$.

   (c) $f(x) = \frac{x}{1-x^2}$, $-1 < x < 1$.

4. Prove that if f is increasing, then so is f^{-1}, and similarly for decreasing functions.

5. Prove that if f and g are one-one, then $f \circ g$ is also one-one. Find $(f \circ g)^{-1}$ in terms of f^{-1} and g^{-1}.

6. Find g^{-1} in term of f^{-1} if $g(x) = 1 + f(x)$.

7. Show that $f(x) = \frac{ax+b}{cx+d}$ is one-one if and only if $ad - bc \neq 0$, and find f^{-1} in this case.

8. On which interval $[a, b]$, will the function $f(x) = x^3 - 3x^2$ be one-one?

9. Suppose that f is differentiable with derivative $f'(x) = (1 + x^3)^{-1/2}$. Show that $g = f^{-1}$ satisfies $g''(x) = \frac{3}{2}g(x)^2$.

10. Suppose that f is a one-one and continuous function and that f^{-1} has a derivative which is nowhere 0. Prove that f is differentiable.