

UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 7

21 September, 2012

1. Show that on the graph of any quadratic polynomial $p(x)$, the chord joining the points $(a, p(a))$ and $(b, p(b))$ is parallel to the tangent line at the midpoint $x = (a + b)/2$.
2. Use Rolle's theorem to show that regardless of the value of b , there is at most one point $x \in [-1, 1]$ for which $x^3 - 3x + b = 0$. Hint: What will happen if there are two such values of $x \in [-1, 1]$?
3. Show that the equation $x^2 = x \sin x + \cos x$ has exactly two real roots. Hint: This question has several steps – first, think about what will happen if there are three or more roots.
4. Sketch the graphs of
 - (a) $1/(x - 1)(x - 3)$
 - (b) $(x^2 - 4)/(x^2 - 9)$
5. For $x > 0$, let $f(x) = 5x^2 + Ax^{-5}$, where $A > 0$ is a constant. Find the smallest value of A such that $f(x) \geq 24$ for all $x > 0$.
6. Given n real numbers a_1, a_2, \dots, a_n , let

$$f(x) = \sum_{i=1}^n (x - a_i)^2.$$

Show that the least value of $f(x)$ is attained when x is the arithmetic mean of a_1, a_2, \dots, a_n .

7. Consider the function

$$f(x) = \begin{cases} 1, & x = 1, 1/2, 1/3, 1/4, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Find all local maxima and minima for $f(x)$.

8.
 - (a) Prove that if $f'(x) \geq M$ for all $x \in [a, b]$, then $f(b) \geq f(a) + M(b - a)$.
 - (b) Prove that if $f'(x) \leq m$ for all $x \in [a, b]$, then $f(b) \leq f(a) + m(b - a)$.
 - (c) Formulate a similar result when $|f'(x)| \leq M$ for all $x \in [a, b]$.

9. Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \dots + \frac{a_n}{n+1} = 0,$$

then

$$a_0 + a_1x + \dots + a_nx^n = 0$$

for some $x \in [0, 1]$.

10. Suppose that f satisfies

$$f''(x) + f'(x)g(x) - f(x) = 0$$

for some function g . Prove that if f is 0 at two points then f is 0 on the interval between them.