UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 6

14 September, 2012

1. Find f' if $f(x) = [x]$.

2. Suppose that f is differentiable and periodic with period a, i.e., $f(x+a) = f(x)$ for all x. Prove that f' is also periodic.

3. Suppose that $f(a) = g(a)$ and that the left-hand derivative of f at a equals the right-hand derivative of g at a. Define $h(x) = f(x)$ for $x \leq a$ and $h(x) = g(x)$ for $x \geq a$. Prove that h is differentiable at a.

4. Let $f(x) = x^2$ if x is rational, and $f(x) = 0$ if x is irrational. Prove that f is differentiable at 0.

5. Let f be a function such that $|f(x)| \leq |x|^{1+\delta}$, where $\delta > 0$. Show that f is differentiable at 0.

6. Show that if $f(x) = g(x)$ for all x in some open interval containing a, then $f'(a) = g'(a)$.

7. Find $f'(0)$ if

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

   and $g(0) = g'(0) = 0$.

8. (a) Prove that if f is differentiable at a then $|f|$ is also differentiable at a provided that $f(a) \neq 0$.
   (b) Give a counterexample if $f(a) = 0$.
   (c) Prove that if f and g are differentiable at a then the functions $\max(f, g)$ and $\min(f, g)$ are differentiable at a, provided that $f(a) \neq g(a)$.

9. Suppose that $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. Prove Leibniz's formula:

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a).$$

10. Suppose that $f(x) = xg(x)$ for some function g which is continuous at 0. Prove that f is differentiable at 0, and find $f'(0)$ in terms of g.