UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 5

7 September, 2012

1. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $f(x)$ is always rational. What can be said about f?

2. Prove that there exists a real number x such that
$$\sin x = x - 1.$$

3. Prove that there exists a real number x such that
$$x^{179} + \frac{163}{1 + x^2 + \sin^2 x} = 119.$$

4. Suppose that f is continuous on $[-1, 1]$ such that $x^2 + (f(x))^2 = 1$ for all x. Show that either $f(x) = \sqrt{1 - x^2}$ or else $f(x) = -\sqrt{1 - x^2}$ for all x.

5. Suppose that f and g are continuous, that $f^2 = g^2$, and that $f(x) \neq 0$ for all x. Prove that either $f(x) = g(x)$ or else $f(x) = -g(x)$ for all x.

6. Suppose that f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some $x \in (a, b)$.

7. Let $f : [a, b] \to [a, b]$ be a continuous function. Show that there exists an $x \in [a, b]$ such that $f(x) = x$.

8. Let $f : [a, b] \to [a, b]$ be a continuous function such that $f(x) > 0$ for all $x \in [a, b]$. Show that there exists an $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in [a, b]$.

9. Let $f : [a, b] \to [a, b]$ be a continuous function with the property: for each $x \in [a, b]$ there exists a $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove that their exists a point $c \in [a, b]$ such that $f(c) = 0$.

10. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.