UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 3

24 August, 2012

1. In each of the following cases, determine whether the given sequence converges or diverges. Find the limit in case the sequence converges.

   (a) $a_n = n/(n+1) - (n+1)/n$

   (b) $a_n = \cos(n\pi/2)$

   (c) $a_n = n/3^n$

   (d) $a_n = (-1)^n/n + (1 + (-1)^n)/n$

   (e) $a_n = \frac{n^{9/10}\sin(n^2)}{n+1}$

   (f) $a_n = \log_a n/n$ where $a > 1$

   (g) $a_n = 1 + \frac{n}{n+1}\cos(n\pi/2)$

   (h) $a_n = a^{1/n}$ where $a > 0$

   (i) $a_n = n^{1/n}$

   (j) $a_n = (p^n + q^n)^{1/n}$ where $p, q \geq 0$

   (k) $a_n = n - (n+p)^{1/2}(n+q)^{1/2}$ where p, q are real.

   (l) $a_n = \frac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}$

2. Let $a_n = \frac{2n}{n^3+1}$. Find the limit of the sequence a_n as $n \to \infty$ and call it L. Now for $\epsilon = 0.1, 0.0001$ find a value of N such that

$$|a_n - L| < \epsilon.$$

   Do the same if $a_n = (-9/10)^n$.

3. Assume that

$$\lim_{n\to\infty} a_n = 0.$$

   Using the definition of limit prove that $\lim_{n\to\infty} a_n^2 = 0$.

4. Here is a list of a_n, the n-th term of a series. In each case, decide whether the series

$$\sum a_n$$

   converges and provide a reason for your answer.

   (a) $a_n = \frac{n}{(4n-3)(4n-1)}$

   (b) $a_n = \frac{\sqrt{2n-1}\log(4n+1)}{n(n+1)}$

(c) $a_n = \frac{|\cos(n)|}{n^2}$

(d) $a_n = (n+1)/2^n$

(e) $a_n = n!/(n+2)!$

(f) $a_n = \log n / n(n+1)^{1/2}$

(g) $a_n = 1/(\log n)^s$

(h) $a_n = 1/\{n \log n (\log\log n)^s\}$

(i) $a_n = 1/(500n + 4)$

(j) $a_n = (n!)^2/(2n)!$

(k) $a_n = e^{-n^2}$

(l) $a_n = 3^n n!/n^n$

(m) $a_n = n^{n+1/n}/(n+1/n)^n$

5. Let $\{x_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$. If $l < 1$, show that $\lim_{n\to\infty} x_n = 0$.

6. Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$. Show that $\{x_n\}$ is convergent and find the limit.

7. Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u = \sup A$, i.e., $u = \sup\{x | x \in A\}$. Show that there is a non-decreasing sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = u$.

8. Let $x_n = 1/1^2 + 1/2^2 + \ldots + 1/n^2$. Prove that $\{x_n\}$ converges.

9. Suppose that $\{x_n\}$ is unbounded. Show that there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} \frac{1}{x_{n_k}} = 0$.

10. If $x_1 < x_2$ are arbitrary real numbers and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n > 2$, show that $\{x_n\}$ is convergent. What is its limit?