UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 2

20 August, 2012

1. Sketch the set of all points (x, y) that satisfy each of the following conditions. Note that $[x]$ denotes the largest integer less than or equal to x, while $\{x\}$ denotes the distance of x from the nearest integer.

   (a) $|x + y| < 1$.
   (b) $x + 2y$ is an integer.
   (c) $1/(x + y)$ is an integer.
   (d) $x^2 - 2xy + y^2 = 9$.
   (e) $|x - 1| = |y - 1|$.
   (f) $x = \sin y$.
   (g) $[x]$.
   (h) $\sqrt{x - [x]}$.
   (i) $[1/x]$.
   (j) $\{x\}$.
   (k) $\{x\} + \{2x\}/2$.

2. If you are given the graph of $f(x)$, describe the graph of the following functions: c denotes a fixed real number, while $\max(a, b)$ and $\min(a, b)$ denote the maximum and the minimum of a, b respectively.

   (a) $f(x) + c$.
   (b) $f(cx)$.
   (c) $f(|x|)$.
   (d) $|f(x)|$.
   (e) $\max(f(x), 0)$.
   (f) $\min(f(x), 1)$.

3. Suppose that $|x - x_0| < \epsilon/2$ and $|y - y_0| < \epsilon/2$, show that

$$|(x + y) - (x_0 + y_0)| < \epsilon \text{ and } |(x - y) - (x_0 - y_0)| < \epsilon.$$

4. Suppose that

$$|x - x_0| < \min\left(\frac{\epsilon}{2(|y_0| + 1)}, 1\right) \text{ and } |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)}.$$

   Show that $|xy - x_0y_0| < \epsilon$.

5. Prove that if $y_0 \neq 0$ and

$$|y - y_0| < \min\left(|y_0|/2, \epsilon |y_0|^2/2\right)$$

then $y \neq 0$ and $|1/y - 1/y_0| < \epsilon$.

6. Now combine your conclusions from (4) and (5) above to find conditions on $|x - x_0|$ and $|y - y_0|$ which will guarantee that if $y_0 \neq 0$, then $y \neq 0$ and $|x/y - x_0/y_0| < \epsilon$.

7. Compute the following limits: (the superscripts $\pm$ in the first two limit questions below are meant to indicate one-sided limits. In the remaining two limit questions, you should use the fact that the limit of $\sin x/x$ as x approaches zero is 1 – no other technique may be used at this stage.)

   (a) $\lim_{x\to 0^+} |x|/x$
   (b) $\lim_{x\to 0^-} [1/x]$
   (c) $\lim_{x\to a} (\sin x - \sin a)/(x - a)$
   (d) $\lim_{x\to 0} (1 - \cos x)/x^2$

8. Suppose that g is a function that satisfies

$$\lim_{x\to 0} g(x) = 0.$$

   Show that

$$\lim_{x\to 0} g(x)\sin(1/x) = 0.$$

   Remember that for every $\epsilon > 0$, you must find a $\delta > 0$ such that ....

9. Suppose that $f(x) \leq g(x)$ for all x. Prove that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, provided that these limits exist. If $f(x) < g(x)$, does it necessarily follow that $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$?

10. Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Prove that $\lim_{x\to a} g(x)$ exists, and that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x)$.