UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 14

16 November, 2012

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given below. In each case determine whether T is linear. If T is linear, describe its null space and range, and compute its nullity and rank.

   (a) $T(x, y) = (y, x)$.

   (b) $T(x, y) = (x^2, y^2)$.

   (c) T maps each point onto its reflection with respect to a fixed line through the origin.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given below. In each case determine whether T is linear. If T is linear, describe its null space and range, and compute its nullity and rank.

   (a) $T(x, y, z) = (z, y, x)$.

   (b) $T(x, y, z) = (x, y^2, z^3)$.

   (c) $T(x, y, z) = (x + z, 0, x + y)$.

3. Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformations given by the formulas: $S(x, y, z) = (z, y, x)$, $T(x, y, z) = (x, x + y, x + y + z)$.

   (a) Determine the images of (x, y, z) under each of the following transformations: $ST, TS, ST - TS$.

   (b) Prove that S and T are one-to-one on $\mathbb{R}^3$ and find the images of (u, v, w) under each of the following transformations: $S^{-1}, T^{-1}, (ST)^{-1}, (TS)^{-1}$.

   (c) Find the image of (x, y, z) under $(T - I)^n$ for each $n \geq 1$.

4. Let S, T be two linear transformations such that $ST - TS = I$. Prove that $ST^n - T^nS = nT^{n-1}$ for all $n \geq 1$.

5. Determine the matrix for the projection $T : \mathbb{R}^5 \to \mathbb{R}^3$ where $T(x_1, x_2, x_3, x_4, x_5) = (x_2, x_3, x_4)$.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(e_1 + e_2) = 3e_1 + 9e_2,\ T(3e_1 + 2e_2) = 7e_1 + 23e_2.$$

   (a) Compute $T(e_2 - e_1)$ and determine the nullity and rank of T.

   (b) Determine the matrix of T relative to the given (standard) basis.

7. Find all 2×2 matrices such that $A^2 = 0$.

8. Prove that
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad - bc)I.$$
Deduce that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular if and only if $ad - bc \neq 0$, in which case its inverse is
$$\frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

9. Prove or disprove: if A and B are non-singular then $A + B$ is non-singular.

10. If A is a square matrix such that $A^2 = A$, prove that $(A+I)^k = I + (2^k - I)A$.