UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 13

12 November, 2012

1. Find all real t such that the two vectors $(1+t, 1-t)$ and $(1-t, 1+t)$ in $\mathbb{R}^2$ are independent.

2. Let $A = (1,2), B = (2,-4), C = (2,-3)$ and $D = (1,-2)$ be four vectors in $\mathbb{R}^2$. Display all non-empty subsets of $\{A, B, C, D\}$ which are linearly independent.

3. Given three linearly independent vectors A, B, C in $\mathbb{R}^n$, prove or disprove each of the following statements.
   (a) $A + B, B + C, A + C$ are linearly independent.
   (b) $A - B, B + C, A + C$ are linearly independent.

4. Prove that a set S of three vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$ if and only if the linear span $L(S)$ contains the three unit vectors e_1, e_2, e_3.

5. Find two bases for $\mathbb{R}^3$ containing the two vectors $(0,1,1)$ and $(1,1,1)$.

6. Consider the following sets of vectors in $\mathbb{R}^3$:

   $S = \{(1,1,1), (0,1,2), (1,0,-1)\}, T = \{((2,1,0), (2,0,-2)\}, U = \{(1,2,3), (1,3,5)\}.$

   (a) Prove that $L(T) \subset L(S)$.
   (b) Determine all inclusion relations that hold among the sets $L(S), L(T)$ and $L(U)$.

7. Let A and B be two subsets in $\mathbb{R}^n$. Prove each of the following statements.
   (a) If $A \subset B$, then $L(A) \subset L(B)$.
   (b) $L(A \cap B) \subset L(A) \cap L(B)$.
   (c) Give an example in which $L(A \cap B) \neq L(A) \cap L(B)$.

8. Let $\mathbb{R}_+$ be the set of positive real numbers. Define the "sum" of two elements x and y in V to be their product xy (in the usual sense), and define "multiplication" of an element x in V by a scalar $c \in \mathbb{R}$ to be x^c. Prove that V is vector space with 1 as the zero element.

9. Show that the vector space $\mathbb{R}$ over the scalar field $\mathbb{Q}$ is infinite dimensional.

10. Let V be a finite dimensional vector space and S a subspace of V. Prove that S is finite dimensional and $dim(S) \leq dim(V)$.