UM 101 : Analysis and Linear Algebra I

August - December 2012

Indian Institute of Science

Exercises 12

6 November, 2012

1. Make a sketch showing the set of all z in the complex plane which satisfy the given relations.

   (a) $z + \bar{z} = 1$.

   (b) $|z - 1| = |z + 1|$.

   (c) $|z - i| = |z + i|$.

   (d) $z + \bar{z} = |z|^2$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial with real coefficients.

   (a) Show that $\overline{f(z)} = f(\bar{z})$ for every $z \in \mathbb{C}$.

   (b) Use part (a) to deduce that non-real zeros of f (if any exist) must occur in pairs of conjugate complex numbers.

3. Let $w = \frac{az+b}{cz+d}$ where a, b, c and d are real. Prove that

$$w - \bar{w} = \frac{(ad - bc)(z - \bar{z})}{|cz + d|^2}.$$

   If $ad - bc > 0$, prove that the imaginary parts of z and w have the same sign.

4. If $z_i \in \mathbb{C}, i = 1, 2, 3$, are vertices of an isosceles triangle, right angled at the vertex z_2, prove that

$$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3).$$

5. Show that the equation of a straight line in the complex plane can be put in the form $z\bar{b} + b\bar{z} = c$, where $b \in \mathbb{C}$ and $c \in \mathbb{R}$.

6. Let $X_1 = (1, 1, 1), X_2 = (0, 1, 1)$ and $X_3 = (1, 1, 0)$ be three vectors in $\mathbb{R}^3$. Let $X_4 = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$, where $\alpha_i, i = 1, 2, 3$ are scalars.

   (a) Determine the components of X_4.

   (b) If $X_4 = 0$, show that $\alpha_i = 0, i = 1, 2, 3$.

   (c) Find $\alpha_i, i = 1, 2, 3$ if $X_4 = (1, 2, 3)$.

7. If a quadrilateral $OABC$ (O is the origin) in $\mathbb{R}^2$ is a parallelogram having A and C as opposite vertices, prove that $A + \frac{1}{2}(C - A) = \frac{1}{2}B$. What geometrical theorem about parallelogram can you deduce from this equation?

8. Prove or disprove the following statement about vectors in $\mathbb{R}^n$: If $X \cdot Y = X \cdot Z$ and $X \neq 0$, then $Y = Z$.

9. Prove or disprove the following statement about vectors in $\mathbb{R}^n$: If $X \cdot Y = 0$ for every Y then $X = 0$.

10. Prove that for any two vectors A and B in $\mathbb{R}^n$ we have

$$\|A + B\|^2 + \|A - B\|^2 = 2\|A\|^2 + 2\|B\|^2.$$

What geometric theorem about the sides and diagonals of a parallelogram can you deduce from this identity?