\textbf{Karhunen-Loève Expansion}

\textbf{Theorem} Let $\chi_n$ be i.i.d. standard Normal variables. Then, almost surely, the series
\[ W_t := \sum_{n=0}^{\infty} \chi_n \frac{\sqrt{2} \sin(n\pi t)}{n\pi} \]
converges uniformly for $t \in [0, 1]$ and then $W$ is the standard Brownian bridge.

For any fixed $t$, the sequence $\sin(n\pi t)/n\pi$ is in $\ell^2$, and hence the series defining $B_t$ converges a.s. We need the following lemma to prove uniform convergence. It is a weaker form of a famous inequality of Bernstein that asserts that $\|p'\|_{L^\infty} \leq N\|p\|_{L^\infty}$.

\textbf{Lemma} Let $p(t) = \sum_{k=0}^{N-1} c_n \sin(kt)$ (more generally, any trigonometric polynomial of degree at most $N$). Then (i) $\|p'\|_{L^\infty} \leq N^2\|p\|_{L^\infty}$. (ii) There is an interval of length $1/N^2$ on which $|p(t)| \geq \frac{1}{2}\|p\|_{L^\infty}$.

\textbf{Proof} (i) Clearly
\[ \|p'\|_{L^\infty} = \max_{0 \leq s \leq 1} \left| \sum_{n=0}^{N-1} c_n n\pi \sin(n\pi t) \right| \leq \left( \max_{0 \leq n \leq N-1} |c_n| \right) \frac{\pi}{2} N(N-1). \]
By the orthogonality of $\sin(n\pi t)$ on $[0, 1]$, and $\int \sin^2(n\pi t) dt = \frac{1}{2}$, we see that
\[ |c_n| = \frac{1}{2} \left| \int p(t) \sin(n\pi t) dt \right| \leq \frac{1}{2}\|p\|_{L^\infty} \]
from which we get $\|p'\|_{L^\infty} \leq \frac{\pi}{2} N(N-1)\|p\|_{L^\infty} \leq N^2\|p\|_{L^\infty}$.

(ii) Thus, if $|p(t_*)| = \|p\|_{L^\infty}$, then for all $|t-t_*| \leq \frac{1}{2\pi \sqrt{2}}$, part (i) implies that $|p(t) - p(t_*)| \leq \|p'\|_{L^\infty}|t-t_*| \leq \frac{1}{2}\|p\|_{L^\infty}$. Thus $|p(t)| \geq \frac{1}{2}\|p\|_{L^\infty}$ on the interval $[t_* - 1/2N^2, t_* + 1/2N^2]$ which has length $1/N^2$.

\textbf{Proof}[Theorem] Fix $k \geq 1$ and consider $p_k(t) := \sum_{n=2^k \cdot k}^{2^{k+1} - 1} \frac{\sqrt{2} \sin(n\pi t)}{n\pi}$. We would like to get an upper bound for the sup norm of $p_k$. By the lemma, we are assured of an interval of length $2^{-2k-2}$ on which $p_k$ is at least half of $\|p_k\|_{L^\infty}$. Therefore, for any $\lambda > 0$, we get
\[ \int_0^1 \left( e^{\lambda p_k(t)} + e^{-\lambda p_k(t)} \right) dt \geq \frac{1}{2^{2k+2}} e^{\frac{1}{2}\lambda \|p_k\|_{L^\infty}}. \]
Now take expectations over $\chi_n$s to get
\[ \mathbb{E} \left[ e^{\frac{1}{2}\lambda \|p_k\|_{L^\infty}} \right] \leq 2^{2k+2} \int_0^1 \mathbb{E} \left[ e^{\lambda p_k(t)} + e^{-\lambda p_k(t)} \right] dt = 2^{2k+3} \int_0^1 \exp \left\{ \lambda^2 r_k(t) \right\} dt \]
where $r_k(t) = \sum_{n=2^k \cdot k}^{2^{k+1} - 1} \frac{\sin^2(n\pi t)}{n^2 \pi^2}$. By the well known $\mathbb{E}[e^{\alpha X}] = e^{\alpha^2/2}$, Clearly $r_k(t) \leq \frac{1}{\pi^2 \alpha^2}$. Therefore, we get $\mathbb{E} \left[ e^{\frac{1}{2}\lambda \|p_k\|_{L^\infty}} \right] \leq 2^{2k+3} \exp \left\{ \frac{\lambda^2}{\pi^2} \right\}$. By Markov’s inequality, it follows that $\mathbb{P} \left[ \|p_k\|_{L^\infty} \geq x \right] \leq 2^{2k+3} \exp \left\{ \frac{x^2}{\pi^2} - \lambda^2 x \right\}$. With $x = 2^{-k/4}$ and $\lambda = 2^{k/2}$, we get
\[ \mathbb{P} \left[ \|p_k\|_{L^\infty} \geq 2^{-k/4} \right] \leq 2^{2k+3} \exp \left\{ \frac{1}{\pi^2} - 2^{k/4} \right\} \]
which is rapidly decaying in $k$ and hence by Borel Cantelli, we see that almost surely, $\|p_k\|_{L^\infty} \leq 2^{-k/4}$ for all large $k$. This implies that $W_1 = \sum_k p_k(t)$ is uniformly convergent for $t \in [0, 1]$, a.s.

From the uniform convergence it follows that $W$ is a.s. a continuous function on $[0, 1]$. It is also a Gaussian process since $\chi_n$ are i.i.d. Normal. To show that $W$ is the Brownian bridge, it suffices to show that its covariance kernel
\[ \sum_{n=1}^{\infty} \frac{2 \sin(n\pi t) \sin(n\pi s)}{\pi^2 n^2} = \begin{cases} t(1-s) & \text{if } t < s, \\
(1-t) & \text{if } s < t. \end{cases} \]
We showed this in class (try a direct proof!).\[ \square \]