Statistical Mechanics on Sparse Random Graphs

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Outline

1. What is this talk about, and why should one care

2. Interesting phenomena

3. A few results
   - Ferromagnetic Ising model
   - Ising spin glass
   - Trees vs graphs: from reconstruction to pure states

4. Conclusion
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What is this talk about, and why should one care
\[ G = (V, E), \ V = [n], \ x = (x_1, \ldots, x_n), \ x_i \in \mathcal{X} \text{ (finite set).} \]

\[
\mu(x) = \frac{1}{Z} \prod_{(ij) \in E} \psi_{ij}(x_i, x_j).
\]
‘Standard model’ (assumptions)

1. $G$ has bounded degree (on average).

2. $G$ has girth larger than $2\ell$ with $\ell = \ell(n) \to \infty$ (apart from $o(n)$ vertices).

3. $0 \leq \psi_{ij}(x_i, x_j) \leq \psi_{\text{max}} < \infty$. For each $i$ exists $x_i^p$ s.t. $0 < \psi_{\text{min}} \leq \psi_{ij}(x_i^p, x_j)$. 
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   For each \( i \) exists \( x_i^p \) s.t. \( 0 < \psi_{\text{min}} \leq \psi_{ij}(x_i^p, x_j) \).
Example 1: $q$-coloring

$G = (V, E)$ graph.

$x = (x_1, x_2, \ldots, x_n), x_i \in \{1, \ldots, q\}$ variables
Example 1: $q$-coloring

$G = (V, E)$ graph.

$x = (x_1, x_2, \ldots, x_n), \quad x_i \in \{1, \ldots, q\} \quad \text{variables}$
Uniform measure over proper colorings

\[ \mu(x) = \frac{1}{Z} \prod_{(i,j) \in E} \psi(x_i, x_j), \quad \psi(x, y) = \mathbb{I}(x \neq y). \]
Example 2: $k$-satisfiability

$n$ variables: $x = (x_1, x_2, \ldots, x_n)$, $x_i \in \{0, 1\}$

$m$ $k$-clauses

$$(x_1 \lor \overline{x_5} \lor x_7) \land (x_5 \lor x_8 \lor \overline{x_9}) \land \cdots \land (\overline{x_{66}} \lor \overline{x_{21}} \lor \overline{x_{32}})$$
Uniform measure over solutions

\[ F = \cdots \land \left( x_{i_1(a)} \lor \overline{x}_{i_2(a)} \lor \cdots \lor \overline{x}_{i_k(a)} \right) \land \cdots \]

\[ \mu(\bar{x}) = \frac{1}{Z} \prod_{a=1}^{m} \psi_a(x_{i_1(a)}, \ldots, x_{i_k(a)}) \]
Many other examples

- Communications (LDPC; XORSAT).
- Artificial intelligence (Bayesian networks; Graphical models).
- Statistics (Compressed sensing).
- ...
Interesting phenomena
1. ‘Exact’ predictions

Example: Free energy density

\[ Z_n \equiv \sum_x \prod_{(ij) \in E} \psi_{ij}(x_i, x_j) \]

\[ \phi \equiv \lim_{n \to \infty} \frac{1}{n} \log Z_n. \]

[Cavity/Replica methods]
1. ‘Exact’ predictions

Example: Free energy density

\[ Z_n \equiv \sum_{\times} \prod_{(ij) \in E} \psi_{ij}(x_i, x_j) \]

\[ \phi \equiv \lim_{n \to \infty} \frac{1}{n} \log Z_n. \]

[Cavity/Replica methods]
2. Mean field equations

‘Set of $O(n)$ non-linear equations that determine local marginals in the large system limit’
2. Mean field equations

Bethe-Peierls equations (replica symmetric cavity method):

\[ \mu_{i \rightarrow j}(\cdot) \equiv \text{Marginal of } x_i \text{ when replacing } \psi_{ij}(x_i, x_j) \text{ by } 1 \]

\[ \mu_{i \rightarrow j}(x_i) \approx \frac{1}{z_{i \rightarrow j}} \prod_{l \in \partial i \setminus j} \sum_{x_l} \psi_{il}(x_i, x_l) \mu_{l \rightarrow i}(x_l) \]

General philosophy: approximate local marginals of \( \mu(\cdot) \) in terms of measures on trees.
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General philosophy: approximate local marginals of \( \mu(\cdot) \) in terms of measures on trees.
2. Bethe-Peierls approximation

\[ F = (U, E_U) \subseteq G, \text{ diam}(F) \leq 2\ell, \text{ such that } \partial i \in U \text{ or } \partial i \cap U = \{ u(i) \} \]

\[ \mu_U(x_U) \approx \nu_U(x_U) = \frac{1}{Z_U} \prod_{(i,j) \in E_U} \psi_{ij}(x_i, x_j) \prod_{i \in \partial U} \nu_{i \rightarrow u(i)}(x_i). \]
\{\nu_{i \rightarrow j}(\cdot)\} \rightarrow \text{‘set of messages’ (aka cavity fields)}

1. Is \(\mu(\cdot)\) well-approximated by some \(\{\nu_{i \rightarrow j}(\cdot)\}\)?

2. How to find a good set of messages \(\{\nu_{i \rightarrow j}(\cdot)\}\)?
Questions

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\{\nu_{i \to j}(\cdot)\} \rightarrow \text{‘set of messages’ (aka cavity fields)}

1. Is \(\mu(\cdot)\) well-approximated by some \(\{\nu_{i \to j}(\cdot)\}\)?

2. How to find a good set of messages \(\{\nu_{i \to j}(\cdot)\}\)?
3. ‘Dynamical’ phase transition

‘The free energy density is analytic but the measure $\mu$ splits into lumps’
3. ‘Dynamical’ phase transition

Example: $k$-satisfiability, the space of solutions

$\alpha_d(k)$ \hspace{1cm} $\alpha_c(k)$ \hspace{1cm} $\alpha_s(k)$

$\alpha = \frac{m}{n}$ fixed, $n \to \infty$.

[Biroli, Monasson, Weigt 00, Mézard, Parisi, Zecchina 02, Krzákala et al 07]
3. ‘Dynamical’ phase transition

Example: $q$-COL, the space of solutions

$\gamma_d(q)$, $\gamma_c(q)$, $\gamma_s(q)$

[Edges taken independently with probability $\gamma/n$ each, $\gamma$ fixed, $n \to \infty$]

[same references + Achlioptas, Ricci 06]
4. ‘Non-self averaging’

\[ x^{(1)}, x^{(2)} \] independent configurations, same disorder \((\text{replicas})\)

\[ d(x^{(1)}, x^{(2)}) \] Hamming distance

\[ \mu(d(x^{(1)}, x^{(2)}) > n\delta) \rightarrow \text{non-degenerate random variable} \]

[\sim \text{SK model}]
4. ‘Non-self averaging’

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[\sim SK model]
A few results
Ferromagnetic Ising model
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\[ G_n = (V_n \equiv [n], E_n) \]

\[ x_i \in \{+1, -1\} \]

\[ \beta \geq 0 \]

\[ \mu(x) = \frac{1}{Z} \exp \left\{ \beta \sum_{(ij) \in E_n} x_ix_j + B \sum_{i=1}^{n} x_i \right\} \]

[in sparse random graphs: Johnston, Plechác 98/ Leone et al 04]
Ferromagnetic Ising model

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[in sparse random graphs: Johnston, Plechác 98/ Leone et al 04]
Theorem (D., Montanari, 10)

If \( \{G_n\} \) is uniformly sparse and converges locally to MGW \( T(P, \rho, \infty) \), then

\[
\phi = \phi^*_N(P, \beta, B) .
\]

[moment condition relaxed in Dommers, Giardina, van der Hofstad 10; extended to all limiting trees and to ferromagnetic Potts models with regular limiting tree in D., Montanari, Sly, Sun, 11]
Theorem (D., Montanari, 10)

If \( \{G_n\} \) is uniformly sparse and converges locally to MGW \( T(P, \rho, \infty) \), then

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\phi = \phi_*(P, \beta, B).
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[moment condition relaxed in Dommers, Giardina, van der Hofstad 10; extended to all limiting trees and to ferromagnetic Potts models with regular limiting tree in D., Montanari, Sly, Sun, 11]
$T(P, \rho, t)$

$P_k$

$\rho_k$
$T(P, \rho, t)$
$T(P, \rho, t)$
$T(P, \rho, t)$
$T(P, \rho, t)$
’Converges locally’

\[ P \equiv \{P_k\}_{k \geq 0} \] Degree distribution, law of \( L \) (of mean \( \bar{P} > 0 \))

\[ \rho \equiv \{\rho_k\}_{k \geq 0} \] Size-biased \( P \), law of \( K \) (degree of uniform edge)

\( T(P, \rho, t) \) \( t \)-generations GW tree (root degree \( P \), else \( \rho \))

\( B_i(t) \) Ball of radius \( t \) in \( G_n \) centered at node \( i \)

**Definition**

\( \{G_n\} \) converge locally to \( T(P, \rho, \infty) \) \( T(P, \rho, \infty) \) if for uniformly random \( I \in [n] \) and fixed \( t \), law of \( B_I(t) \) converges as \( n \to \infty \) to \( T(P, \rho, t) \).

[in framework of Benjamini & Schramm 01, Aldous & Lyons 07]
$\phi_*(P, \beta, B)$

For $B \geq 0$, let $\theta \equiv \tanh \beta$, $h^{(0)} > 0$, and for iid $h_i^{(t)}$,

$$h^{(t+1)} \overset{d}{=} \tanh \left\{ B + \sum_{i=1}^{K-1} \text{atanh}(\theta h_i^{(t)}) \right\},$$

Then $h^{(t)} \overset{d}{\rightarrow} h^*$ and for iid $h_i^*$ independent of $L$,

$$\phi_*(P, \beta, B) = \log \cosh B + \frac{P}{2} \log \cosh \beta - \frac{P}{2} \mathbb{E} \log (1 + \theta h_1^* h_2^*) +$$

$$+ \mathbb{E} \log \left\{ (1 + \tanh B) \prod_{i=1}^{L} (1 + \theta h_i^*) + (1 - \tanh B) \prod_{i=1}^{L} (1 - \theta h_i^*) \right\}.$$

[Variational (LD) formulation for $\phi_*$, see D., Montanari, Sun, 11]
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[Variational (LD) formulation for $\phi_{*}$, see D., Montanari, Sun, 11]
0. Take $B > 0$.

1. Reduce to expectations of local quantities

$$\frac{d}{d\beta} \log Z_n(\beta, B) = \sum_{(ij) \in E_n} \langle x_i x_j \rangle_n$$

($\langle \cdot \rangle_n$ denote expectation under Ising on $G_n$).

2. Prove convergence of local expectations to tree values.
2. Convergence to tree values

- $\mathcal{T}$: infinite tree with max degree $k_{\text{max}}$
- $\mathcal{T}(t)$: first $t$ generations
- $\mu_t^{t,z}(\cdot)$: Ising model on $\mathcal{T}(t)$ boundary condition $z$
- $\mu_{r,t}^{t,z}$: root spin expectation
2. Convergence to tree values

\[ \mathcal{T} \text{ infinite tree with max degree } k_{\text{max}} \]

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\[ \mu_{r}^{t,z} \text{ root spin expectation} \]
2. Convergence to tree values

\[ \mathcal{T} \] infinite tree with max degree \( k_{\text{max}} \)

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\[ \mu_{t,z}(\cdot) \] Ising model on \( \mathcal{T}(t) \) boundary condition \( z \)

\[ \mu_{r,z} \] root spin expectation
Uniform (Gibbs measure uniqueness)

\[ |\mu^{t,z(1)}_r - \mu^{t,z(2)}_r| \leq |\mu^{t,\pm}_r - \mu^{t,-}_r| \to 0. \]

Easier

True only at high temperature \((\beta = O(1/k_{max}))\)
Uniform (Gibbs measure uniqueness)

\[ |\mu^t_{r,z(1)} - \mu^t_{r,z(2)}| \leq |\mu^t_{r,+} - \mu^t_{r,-}| \to 0. \]
Uniform (Gibbs measure uniqueness)

\[ |\mu_{r, z(1)}^t - \mu_{r, z(2)}^t| \leq |\mu_{r, +}^t - \mu_{r, -}^t| \to 0. \]

Easier

True only at high temperature \((\beta = O(1/k_{\text{max}}))\)
For $\beta > \beta_c \equiv \text{atanh}(1/\bar{\rho})$

$$\lim_{B \to 0^+} \lim_{n \to \infty} \mathbb{E}\langle x_I \rangle_n = - \lim_{B \to 0^-} \lim_{n \to \infty} \mathbb{E}\langle x_I \rangle_n > 0$$
\( z = (+1, +1, \ldots, +1) \; \Rightarrow \; \lim_{\ell \to \infty} \langle x_r \rangle_\ell > 0 \)

\( z = (-1, -1, \ldots, -1) \; \Rightarrow \; \lim_{\ell \to \infty} \langle x_r \rangle_\ell < 0 \)
Non uniform

\[ | \mu_{r,}^{t,+} - \mu_{r,}^{t,\text{free}} | \rightarrow 0. \]
0 \leq \mu^t_r, + - \mu^t_r, free \leq \epsilon\{\mu^t_r, free - \mu^{t-1}_r, free\} \to 0

(\mu^t_r, free \text{ monotone by Griffiths})

[Ising specific, but strategy extended to Potts, Independent Sets]
Ising spin glass
Ising spin glass

Statistical Mechanics on Sparse Random Graphs
Ising spin glass

\[ G_n = (V_n \equiv [n], E_n) \]

\[ x_i \in \{+1, -1\} \]

\[ \mu(x) = \frac{1}{Z} \exp \left\{ \beta \sum_{(ij) \in E_n} J_{ij} x_i x_j + B \sum_{i=1}^{n} x_i \right\} \]

\[ J_{ij} \in \{+1, -1\} \text{ uniformly random} \]

[Viana, Bray 1985]
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[Viana, Bray 1985]
Theorem

If $G_n$ uniformly random with average degree $\gamma$ and $\beta < \beta_*(B, \gamma)$, then

$$\phi = \hat{\phi}_*(\gamma, \beta, B).$$

Guerra, Toninelli 2003

$B = 0$, $\beta_* = \text{atanh}(1/\sqrt{\gamma})$

Talagrand 2001, 2003

$B \neq 0$, $\beta_* = O(1/\gamma)$
Theorem

If $G_n$ uniformly random with average degree $\gamma$ and $\beta < \beta^*_B(B, \gamma)$, then

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Free energy density: brief survey

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$B \neq 0, \beta_* = O(1/\gamma)$
Theorem (D., Gerschenfeld, Montanari 11)

If $G_n$ uniformly sparse and converges locally to $T(P, \rho, \infty)$ and $\beta < \beta^*_\infty(B, P)$, then

$$\phi = \hat{\phi}^*_\infty(P, \beta, B).$$

$$K \sim k_{\text{typ}} \gg 1 \Rightarrow \beta^*_\infty(B, P) \sim \frac{f(B)}{\sqrt{k_{\text{typ}}}} \text{ and } 0 < f(B) \uparrow \infty \text{ with } B.$$
Free energy density

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and $0 < f(B) \uparrow \infty$ with $B$. 

Amir Dembo
Statistical Mechanics on Sparse Random Graphs
For iid $\theta_i \in \{+\tanh \beta, -\tanh \beta\}$ uniformly at random, independent of $K$, $L$ and iid $h_i^{(t)}$, let

$$h^{(t+1)} \overset{d}{=} \tanh \left\{ B + \sum_{i=1}^{K-1} \operatorname{atanh}(\theta_i; h_i^{(t)}) \right\},$$

Then $h^{(t)} \overset{d}{\to} h^*$ and for iid $h_i^*$ independent of $L$,

$$\hat{\phi}_*(P, \beta, B) = \log \cosh B + \frac{P}{2} \log \cosh \beta - \frac{P}{2} \mathbb{E} \log (1 + \theta_0 h_1^* h_2^*) +$$

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Trees vs graphs: from reconstruction to pure states
Alice, Bob and $G$
Alice samples a proper coloring (uniformly)
...and hides a ball $B(\text{root}, t)$
Amir Dembo  
Statistical Mechanics on Sparse Random Graphs
...guesses right!

![Diagram of a sparse random graph with a designated root node]
The problem

Does Bob have a chance?
Formally

\[ X = \{X_i : i \in V\} \text{ uniformly random proper coloring.} \]

\[ \mu_U(\cdot | G) \text{ distribution of } X_U \equiv \{X_i : i \in U \subseteq V\} \]

\[ \overline{B}(r,t) = \{i \in V : d(i,r) \geq t\} \]

**Definition**

The reconstruction problem is solvable for the sequence of random rooted graphs \( G_n = (V_n = [n], E_n) \) if for some \( \varepsilon > 0, \)

\[ \| \mu_{r, \overline{B}(r,t)}(\cdot, \cdot | G_n) - \mu_r(\cdot | G_n)\mu_{\overline{B}(r,t)}(\cdot | G_n) \|_{TV} \geq \varepsilon, \]

with positive probability (bounded away from 0 as \( n \to \infty \)).
When $G = \text{Tree}$


→ Evans, Kenyon, Peres, Schulman (2000): Ising on general trees

→ Mossel, Peres (2003): Non binary variables


→ Chayes et al. (2006): Asymmetric Ising.
Pure states decomposition in \(q\)-COL

**Theorem**

\[
\gamma_d(q) \equiv \begin{cases} 
\text{Non-extremality (a.k.a. multiple pure states or dRSB)} \\
(1) \text{ Tree reconstruction threshold} \\
(2) \text{ Graph reconstruction threshold}
\end{cases}
\]

[Conjectured by Mézard, Montanari 06 that also prove (1) for regular tree-like graphs; (2) proved for Erdős-Rényi graphs by Montanari, Restrepo and Tetali 2011]
Combinatorics/Probability problems on random sparse graphs.

Unifying approach: approximation by trees.

Naturally leads to many interesting problems.
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Many challenges

1. General models and trees (e.g. ferromagnetic Potts for general limiting tree).

2. Ising spin glass - push $\beta_*(0, P)$ to the RSB point.

3. Rigorous understanding of the 'one-step replica symmetry' phase (as in $q$-coloring beyond $\gamma_d(q)$) and beyond.
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1. General models and trees (e.g. ferromagnetic Potts for general limiting tree).

2. Ising spin glass - push $\beta_*(0, P)$ to the RSB point.

3. Rigorous understanding of the 'one-step replica symmetry' phase (as in $q$-coloring beyond $\gamma_d(q)$) and beyond.

