CORRECTION TO "A SMOOTH FOLIATION OF THE 5-SPHERE BY COMPLEX SURFACES".

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Our paper [M-V] claims that it describes a foliation of $\mathbb{S}^5$ by complex surfaces. However it was pointed out to us by the anonymous referee of a related article that the foliation constructed in the paper lives in fact on a 5-manifold with non-trivial fundamental group. The aim of this note is to explain this difference and to characterize this 5-manifold.

We observe that, even with this modification, this foliation is still the first example of such an exotic CR-Structure. Quoting [M-V2], "We would like to emphasize that, as far as we know, the foliation described in [M-V] (as well as the related examples of [M-V], Section 5) is the only known example of a smooth foliation by complex manifolds of complex dimension strictly greater than one on a compact manifold, which is not obtained by classical methods such as the one given by the orbits of a locally free smooth action of a complex Lie group, the natural product foliation on $M \times N$ where M is foliated by Riemann surfaces and N is a complex manifold, holomorphic fibrations, or trivial modifications of these examples such as cartesian products of known examples or pull-backs. Of course, it is very easy to give examples of foliations by complex manifolds on open manifolds (in fact even with Stein leaves). On the other hand, if a compact smooth manifold has an orientable smooth foliation by surfaces then, using a Riemannian metric and the existence of isothermal coordinates, we see that the foliation can be considered as a foliation by Riemann surfaces."

We use the notations and results of [M-V] and assume that the reader is acquainted with them.

The foliation of [M-V] is obtained by gluing, thanks to Lemma 1, two tame foliations on manifolds with boundary. The first one, $\mathcal{M}$ is a bundle over the circle with fiber the affine Fermat surface

$$F = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \quad | \quad P(z) = z_1^3 + z_2^3 + z_3^3 = 1\}$$

and monodromy

$$z \in F \longmapsto \omega \cdot z \in F$$

where $\omega = \exp(2i\pi/3)$. Its foliation is described in [M-V, Section 3].

The second one, $\mathcal{N}$, is supposed to be diffeomorphic to $K \times \overline{\mathbb{D}}$ (where K is a circle bundle over a torus, see [M-V, Section 1.2]). Its foliation is given as a quotient foliation: take the quotient of

$$\tilde{X} = \mathbb{C}^* \times (\mathbb{C} \times [0, \infty) \setminus \{(0,0)\}),$$

endowed with the trivial foliation by the level sets of the $[0, \infty)$-coordinate, by the abelian group generated by

$$\begin{cases} T(z, u, t) = (\exp(2i\pi\omega) \cdot z, ((\psi(z))^{-3} \cdot u, t) \\ U(z, u, t) = (z, \exp(2i\pi\tau) \cdot u, d(t)) \end{cases}$$

(cf. [M-V, Section 2] for the definition of ψ and d). The map T is chosen such that the quotient space of the boundary of $\tilde{X}$ (corresponding to $t = 0$) by the T-action is a particular $\mathbb{C}^*$ bundle of Chern class -3 over the elliptic curve $\mathbb{E}_\omega$, called W in [M-V].

Consider the interior of this manifold with boundary (that is take $t > 0$). The quotient of a leaf $\{t = Constant\}$ by the T-action is biholomorphic to L, the line bundle over $\mathbb{E}_\omega$ associated to W. Hence the quotient of Int $\tilde{X}$ by the T-action is CR-isomorphic to $L \times (0, \infty)$. Now, using the fact that d is contracting, and the fact that the map

$$u \in \mathbb{C} \longmapsto \exp(2i\pi\tau) \cdot u \in \mathbb{C}$$

is isotopic to the identity, we see that the complete quotient is a bundle over the circle with fiber L and monodromy isotopic to the identity, that is CR-isomorphic to $L \times \mathbb{S}^1$. Indeed, this means that $\mathcal{N}$ (if we decide from now on to call $\mathcal{N}$ the previous quotient manifold with boundary) is diffeomorphic to $D \times \mathbb{S}^1$, where D is the $\overline{\mathbb{D}}$-bundle associated to L. It is definitely not diffeomorphic to $K \times \overline{\mathbb{D}}$, since this last manifold has a nilpotent fundamental group (see [M-V, Section 1.2]), whereas $D \times \mathbb{S}^1$ retracts on $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

When gluing $\mathcal{M}$ and this "new" $\mathcal{N}$, one does not obtain the 5-sphere but the following manifold, let us call it Z. Observe that, in the projective Fermat surface

$$F^p = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3 \quad | \quad z_1^3 + z_2^3 + z_3^3 = z_0^3\},$$

the elliptic curve at infinity has a tubular neighborhood diffeomorphic to D. Observe also that the gluing of $\mathcal{M}$ and $\mathcal{N}$ respect the fibrations over the circle, that is the two bases are identified and the gluing occur on the fibers. It follows from all that that Z is a bundle over the circle with fiber F^p and monodromy

$$[z_0 : z_1 : z_2 : z_3] \in F^p \longmapsto [z_0 : \omega \cdot z_1 : \omega \cdot z_2 : \omega \cdot z_3] \in F^p$$

So finally, what is really proved in [M-V] is the following Theorem.

Theorem. *Let Z be the 5-dimensional bundle over the circle with fiber the projective Fermat surface and monodromy the multiplication by the root of unity ω on the affine part.*

There exists on Z an exotic smooth, codimension-one, integrable and Levi-flat CR-structure on Z. The induced foliation by complex surfaces satisfies:

(i) There are only two compact leaves both biholomorphic to an elliptic bundle over the elliptic curve $\mathbb{E}_\omega$. Since this surface has odd first Betti number it is not Kähler.

(ii) One compact leaf is the boundary of a compact set in Z whose interior is foliated by line bundles over $\mathbb{E}_\omega$ with Chern class -3 obtained from W by adding a zero section. The two compact leaves are the boundary components of a collar and the leaves in the interior of this collar are infinite cyclic coverings of the compact leaves and biholomorphic to W, thus they are principal $\mathbb{C}^*$-bundles over the elliptic curve $\mathbb{E}_\omega$.

(iii) The other leaves have the homotopy type of a bouquet of eight copies of $\mathbb{S}^2$ and they are all biholomorphic to the affine complex smooth manifold $P^{-1}(z)$, $z \in \mathbb{C}^*$.

Remarks.
a) Since the manifold Z fibres over the circle with fibre a nonsigular cubic surface, it has a natural foliation by complex leaves given by the fibres. Ours is obviously completely different.
b) Using the polynomials

$$P(z) = z_1^2 + z_2^4 + z_3^4 \qquad \text{resp.} \qquad P(z) = z_1^2 + z_2^3 + z_3^6$$

instead of the cubic one, the construction can be adapted as described in [M-V] to obtain exotic integrable CR-structures on bundles over the circle with fiber

$$F^p = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3 \quad | \quad z_0^2 z_1^2 + z_2^4 + z_3^4 = z_0^4\},$$

and respectively

$$F^p = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3 \quad | \quad z_0^4 z_1^2 + z_0^3 z_2^3 + z_3^6 = z_0^6\}.$$

c) Due to the compact non Kähler leaves, this CR-structure is not embeddable in any Stein space nor Kähler manifold. Moreover, it is not embeddable in any 3-dimensional complex manifold [DS].
d) In [De], G. Deschamps proved that the use of a collar can be avoided by choosing carefully the holonomy of the boundaries of $\mathcal{M}$ and $\mathcal{N}$. This gives a foliation on Z with the same properties as above, but with a single compact leaf.

References

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