REAL THETA CHARACTERISTICS AND AUTOMORPHISMS OF A REAL CURVE

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Abstract. Let X be a geometrically irreducible smooth projective curve, defined over $\mathbb{R}$, of genus at least two that admits nontrivial automorphisms. Fix a nontrivial automorphism σ of X. Assume that X does not have any real points. Then σ acts trivially on the set of all real theta characteristics of X if and only if X is hyperelliptic with σ being the unique hyperelliptic involution of X. Examples are given showing that the condition that X does not have any real points is necessary.

1. Introduction

Let X be a smooth complex projective curve of genus g, with $g \geq 2$. Assume that X admits nontrivial automorphisms; fix a nontrivial automorphism σ of X. In [3] it was shown that if σ fixes pointwise all the theta characteristics of X, then X is hyperelliptic with σ being the unique hyperelliptic involution of X.

Our aim here is to address a similar question for curves defined over the field of real numbers.

Let X be a geometrically irreducible smooth projective curve defined over $\mathbb{R}$. The genus of X, which will be denoted by g, is assumed to be at least two. Let $\text{Pic}^{g-1}(X)_{\mathbb{R}}$ denote the real points of the Picard variety $\text{Pic}^{g-1}(X)$. The real theta characteristics of X are all those points of $\text{Pic}^{g-1}(X)_{\mathbb{R}}$ which are square–roots of the real point of $\text{Pic}^{2g-2}(X)$ given by the cotangent line bundle K_X of X. Therefore, any automorphism of X acts on the set of real theta characteristics of X. The set of real theta characteristics of X will be denoted by $\mathcal{S}(X)$. It should be mentioned that in general not all points of $\mathcal{S}(X)$ represent some real algebraic line bundle over X.

Our first result is the following (see Theorem 2.3):

Theorem 1.1. *Assume that X does not have any real points. Assume that X admits a nontrivial automorphism; fix a nontrivial automorphism σ of X. Then σ acts trivially on $\mathcal{S}(X)$ if and only if X is hyperelliptic with σ being the unique hyperelliptic involution of X.*

The condition in Theorem 1.1 that X does not have any real points is necessary. In other words, there are examples of X with real point, and $\sigma \in \text{Aut}(X)\setminus\{\text{Id}_X\}$, such that

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- the action of σ on $\mathcal{S}(X)$ is trivial, and
- $X/\langle\sigma\rangle \neq \mathbb{P}^1_{\mathbb{R}}$.

See Section 3 for the details.

We also complete the case of odd theta characteristics which was left out in [3]. More precisely, we prove the following (see Proposition 4.2):

Proposition 1.2. *Let Y be a connected smooth complex projective curve of genus g, with $g \geq 2$, that admits a nontrivial automorphism. Fix a nontrivial automorphism σ of Y. If σ fixes pointwise all the odd theta characteristics of Y, then Y is hyperelliptic with σ being the unique hyperelliptic involution of Y.*

It was noted in [3] that the corresponding statement for even theta characteristics on X is valid; see [3, Proposition 2.3].

2. Real theta characteristics

Let X be a geometrically irreducible smooth projective curve defined over the field of real numbers. Let

$$X_{\mathbb{C}} := X \times_{\mathbb{R}} \mathbb{C}$$

be the complexification of X. Therefore, $X_{\mathbb{C}}$ is an irreducible smooth projective curve defined over $\mathbb{C}$. In other words, $X_{\mathbb{C}}$ is a compact connected Riemann surface. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ acts on $X_{\mathbb{C}}$ through an antiholomorphic involution

$$\tau : X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}} . \tag{2.1}$$

The real points of X are precisely the fixed points of this involution τ.

Given a holomorphic line bundle L on the Riemann surface $X_{\mathbb{C}}$, let $\overline{L}$ denote the C^∞ complex line bundle on $X_{\mathbb{C}}$ whose underlying smooth real vector bundle of rank two is identified with the smooth real vector bundle of rank two two underlying L, while the complex structure of a fiber $\overline{L}_x$, $x \in X_{\mathbb{C}}$, is the conjugate of the complex structure of the fiber L_x. The pull back $\tau^*\overline{L}$ has a natural holomorphic structure, where τ is the antiholomorphic involution of $X_{\mathbb{C}}$ in eqn. (2.1). A smooth section of $\tau^*\overline{L}$ defined over an analytic open subset $U \subset X_{\mathbb{C}}$ is holomorphic if and only if the corresponding section of L over $\tau(U)$ is holomorphic.

For any $d \in \mathbb{Z}$, let $\mathrm{Pic}^d(X)$ be the Picard variety of X for degree d. The set of real points of $\mathrm{Pic}^d(X)$ will be denoted by $\mathrm{Pic}^d(X)_{\mathbb{R}}$. We note that $\mathrm{Pic}^d(X)_{\mathbb{R}}$ parametrizes all holomorphic line bundles L of degree d over $X_{\mathbb{C}}$ such that $\tau^*\overline{L}$ is holomorphically isomorphic to L, where τ is defined in eqn. (2.1).

The real algebraic line bundles over X of degree d form a subset of $\mathrm{Pic}^d(X)_{\mathbb{R}}$. However in general not every point of $\mathrm{Pic}^d(X)_{\mathbb{R}}$ corresponds to some real algebraic line bundle over X of degree d. To explain it, let ξ be a real algebraic line bundle of degree d over X. Then the base change $\xi_{\mathbb{C}} := \xi \bigotimes_{\mathbb{R}} \mathbb{C}$ is a holomorphic line bundle over $X_{\mathbb{C}}$ of degree d.

The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ acts on $\xi_{\mathbb{C}}$ through an isomorphism of holomorphic line bundles

$$f : \xi \longrightarrow \tau^*\overline{\xi}$$

such that the composition

$$\xi \xrightarrow{f} \tau^*\overline{\xi} \xrightarrow{\tau^*\overline{f}} \tau^*\overline{\tau^*\overline{\xi}} = \xi$$

is the identity automorphism of ξ. Conversely, any pair $(\zeta\, , f_0)$, where ζ is a holomorphic line bundle over $X_{\mathbb{C}}$ of degree d, and

$$f_0 : \zeta \longrightarrow \tau^*\overline{\zeta}$$

is a holomorphic isomorphism of line bundles such that the composition

$$\zeta \xrightarrow{f_0} \tau^*\overline{\zeta} \xrightarrow{\tau^*\overline{f_0}} \tau^*\overline{\tau^*\overline{\zeta}} = \zeta$$

is the identity automorphism of ζ, define a real algebraic line bundle over X of degree d. The point to note is that in general there are examples of holomorphic line bundles ζ over $X_{\mathbb{C}}$ such that $\tau^*\overline{\zeta}$ is holomorphically isomorphic to ζ, but there is no isomorphism f_0 from ζ to $\tau^*\overline{\zeta}$ that satisfies the condition

$$\tau^*\overline{f_0} \circ f_0 = \mathrm{Id}_{\zeta}\, .$$

(See [4] for more details.)

Let g denote the genus of X. We recall below the definition of a real theta characteristic on X; see [4, p. 167].

Definition 2.1. A point of $\mathrm{Pic}^{g-1}(X)_{\mathbb{R}}$ is called a *real theta characteristic* of X if the corresponding line bundle L over $X_{\mathbb{C}}$ satisfies the condition that $L^{\otimes 2} \cong K_{X_{\mathbb{C}}}$, where $K_{X_{\mathbb{C}}}$ is the cotangent line bundle of the complex curve $X_{\mathbb{C}}$. The set of all real theta characteristics on X will be denoted by $\mathcal{S}(X)$.

Remark 2.2. In the above definition we do not demand that a real theta characteristic of X is a real algebraic line bundle on X. Therefore, the above definition of a real theta characteristic differs from the one given in [6]. However, it coincides with the definition in [4].

Let $\mathrm{Pic}^0(X_{\mathbb{C}})_2 \subset \mathrm{Pic}^0(X_{\mathbb{C}})$ be the subgroup defined by all line bundles of order two. Define

$$\mathcal{A} := \mathrm{Pic}^0(X)_{\mathbb{R}} \cap \mathrm{Pic}^0(X_{\mathbb{C}})_2 \tag{2.2}$$

to be the intersection.

We know that $\mathcal{S}(X)$ is nonempty [2, p. 61], [4, p. 164, Corollary 4.3]. The set $\mathcal{S}(X)$ is evidently a principal homogeneous space (also called a torsor) for $\mathcal{A}$ defined in eqn. (2.2). In other words, the group $\mathcal{A}$ acts freely transitively on the set $\mathcal{S}(X)$. The action of any $\xi \in \mathcal{A}$ on $\mathcal{S}(X)$ is defined by $L \longmapsto L \bigotimes \xi$.

The group $\text{Pic}^0(X_{\mathbb{C}})_2$ in eqn. (2.2) is canonically identified with $H^1(X,\, \mathbb{Z}/2\mathbb{Z})$. So see this, first note that

$$H^1(X,\, \mathbb{Z}/2\mathbb{Z}) \,=\, \text{Hom}(\pi_1(X_{\mathbb{C}}), \mathbb{Z}/2\mathbb{Z})\,.$$

A homomorphism $\alpha \,:\, \pi_1(X_{\mathbb{C}}) \,\longrightarrow\, \mathbb{Z}/2\mathbb{Z})$ gives a flat complex line bundle over $X_{\mathbb{C}}$ of order two, in particular, α gives a holomorphic line bundle over $X_{\mathbb{C}}$ of order two. This identifies the group $\text{Pic}^0(X_{\mathbb{C}})_2$ with $H^1(X,\, \mathbb{Z}/2\mathbb{Z})$.

The involution τ in eqn. (2.1) gives an involution

$$\widehat{\tau} \,:\, H^1(X,\, \mathbb{Z}/2\mathbb{Z}) \,\longrightarrow\, H^1(X,\, \mathbb{Z}/2\mathbb{Z}) \tag{2.3}$$

defined by $\alpha \,\longmapsto\, \tau^*\alpha$. The above identification of the group $\text{Pic}^0(X_{\mathbb{C}})_2$ with $H^1(X,\, \mathbb{Z}/2\mathbb{Z})$ sends the subgroup $\mathcal{A} \,\subset\, \text{Pic}^0(X_{\mathbb{C}})_2$ in eqn. (2.2) surjectively to the invariant subgroup

$$H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}} \,\subset\, H^1(X,\, \mathbb{Z}/2\mathbb{Z}) \tag{2.4}$$

on which $\widehat{\tau}$ coincides with the identity map.

We noted earlier that $\mathcal{S}(X)$ is a principal homogeneous space for $\mathcal{A}$. Therefore, using the above identification $\mathcal{A} \,=\, H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$, we have an action of the group $H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$ on $\mathcal{S}(X)$,

$$H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}} \times \mathcal{S}(X) \,\longrightarrow\, \mathcal{S}(X)\,, \tag{2.5}$$

which makes $\mathcal{S}(X)$ a principal homogeneous space for $H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$.

Given any automorphism σ of the real curve X, let $\widehat{\sigma}$ be the corresponding automorphism of $X_{\mathbb{C}}$. Since $\widehat{\sigma}$ is induced by an automorphism of X, it follows immediately that $\widehat{\sigma}$ commutes with the antiholomorphic involution τ in eqn. (2.1). This implies that for any $L \,\in\, \mathcal{S}(X) \,\subset\, \text{Pic}^{g-1}(X_{\mathbb{C}})$, the pull back $\widehat{\sigma}^*L$ is also an element of $\mathcal{S}(X)$. Therefore, σ acts on the set $\mathcal{S}(X)$ by sending any L to $\widehat{\sigma}^*L$.

Since τ and $\widehat{\sigma}$ commute, the subspace $H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$ in eqn. (2.4) is left invariant by the action of σ on $H^1(X,\, \mathbb{Z}/2\mathbb{Z})$ that sends any $\beta \,\in\, H^1(X,\, \mathbb{Z}/2\mathbb{Z})$ to $\widehat{\sigma}^*\beta$. The map ϕ in eqn. (2.5) is clearly equivariant for the actions of σ on $\mathcal{S}(X)$ and $H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}} \times \mathcal{S}(X)$; the action of σ on $\mathcal{S}(X)$ is defined by $L \,\longmapsto\, \widehat{\sigma}^*L$, and σ acts diagonally on $H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}} \times \mathcal{S}(X)$ through the above actions on $H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$ and $\mathcal{S}(X)$.

Theorem 2.3. *Let X be a geometrically irreducible smooth projective curve, of genus at least two, defined over the field of real numbers. Assume that X admits a nontrivial automorphism; fix a nontrivial automorphism σ of the curve X. Assume that X does not have any real points. Then σ acts trivially on $\mathcal{S}(X)$ if and only if X is hyperelliptic with σ being the unique hyperelliptic involution of X.*

Proof. Let g be the genus of X; so $g \,\geq\, 2$. Since X does not have any real points, the involution τ in eqn. (2.1) does not have any fixed points. Therefore, the quotient

$$Y \,:=\, X_{\mathbb{C}}/\langle\tau\rangle \tag{2.6}$$

is a smooth nonorientable surface. We note that invariant subspace $H^1(X,\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$ in eqn. (2.4) is identified with $H^1(Y,\, \mathbb{Z}/2\mathbb{Z})$, where Y is defined in eqn. (2.6).

Let σ be a nontrivial automorphism of X that acts trivially on $\mathcal{S}(X)$. As before, the corresponding automorphism of $X_{\mathbb{C}} := X \times_{\mathbb{R}} \mathbb{C}$ will be denoted by $\widehat{\sigma}$.

We noted above that the homomorphism ϕ in eqn. (2.5) is equivariant for the actions of σ. We also noted that $\mathcal{S}(X)$ is nonempty, and it is a principal homogeneous space for $H^1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$. Therefore, the given condition that σ acts trivially on $\mathcal{S}(X)$ implies immediately that the action of σ on $H^1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$ defined by $\beta \longmapsto \widehat{\sigma}^*\beta$ is trivial.

Let

$$H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}} \subset H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})$$

be the invariant part of the involution of $H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})$ defined by $\alpha \longmapsto \widehat{\tau}_*(\alpha)$. Similarly, σ acts on $H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})$ by sending any α to $\widehat{\sigma}_*(\alpha)$. Since σ acts trivially on $H^1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$, using the Poincaré duality pairing it follows immediately that σ acts trivially on $H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})^{\widehat{\tau}}$.

For any $\alpha\,,\beta \in H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})$, their cap product, which is an element of $\mathbb{Z}/2\mathbb{Z}$, will be denoted by $\alpha \cdot \beta$. We recall that a linear basis

$$\{\alpha_1\,,\beta_1\,,\cdots\,,\alpha_g\,,\beta_g\} \subset H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})$$

is called *symplectic* if $\alpha_i \cdot \beta_j = \delta_{ij}$ and $\alpha_i \cdot \alpha_j = 0 = \beta_i \cdot \beta_j$ for all i, j. The following lemma will be needed in the proof of the theorem.

Lemma 2.4. *There is a symplectic basis* $\alpha_1\,,\beta_1\,,\cdots\,,\alpha_g\,,\beta_g$ *of* $H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})$ *such that* $\widehat{\tau}_*(\beta_i) = \beta_i$ *for all* $1 \leq i \leq g$.

Proof. The real projective plane and the two–sphere will be denoted by P and S^2 respectively. Let

$$\pi' : S^2 \longrightarrow P$$

be a universal covering map. The topological surface Y in eqn. (2.6) is obtained from P by removing the interiors of g disjoint disks $D_1\,,\cdots\,,D_g$ and gluing in g Möbius bands $M_1\,,\cdots\,,M_g$.

The orientable two–fold cover of the Möbius band is an annulus. The surface $X_{\mathbb{C}}$ is obtained from the sphere S^2 by attaching g disjoint one–handles. Further, the boundary of each of the g Möbius bands $M_i \subset Y$ has inverse image, in $X_{\mathbb{C}}$, a pair of homologous curves. This pair of homologous curves are the boundary components of the corresponding annulus. We note that these two curves are also homologous to the central curve of the annulus. By the standard presentation for the homology of a surface it follows that the central curves of the g annuli form a symplectic half–basis (basis for a Lagrangian subspace) $\beta_1\,,\cdots\,,\beta_g$.

We have seen that each of the elements $\beta_i \in H_1(X_{\mathbb{C}},\, \mathbb{Z}/2\mathbb{Z})$, $i \in [1\,,g]$, is fixed by τ_*. The lemma follows by completing the half–basis $\beta_1\,,\cdots\,,\beta_g$ to a symplectic basis. $\square$

Continuing with the proof of the theorem, we fix a symplectic basis

$$\{\alpha_1\,,\beta_1\,,\cdots\,,\alpha_g\,,\beta_g\}\ \subset\ H_1(X_{\mathbb C},\,\mathbb Z/2\mathbb Z)$$

given by Lemma 2.4; so $\widehat{\tau}_*(\beta_i)\ =\ \beta_i$ for all $1\ \leq\ i\ \leq\ g$. Since σ acts trivially on $H_1(X_{\mathbb C},\,\mathbb Z/2\mathbb Z)^{\widehat{\tau}}$, it follows that

$$\widehat{\sigma}_*(\beta_i)\ =\ \beta_i \tag{2.7}$$

for all $1\ \leq\ i\ \leq\ g$.

Next, observe that for $1\ \leq\ i\ \leq\ g$,

$$\alpha_i\cdot\beta_j\ =\ \widehat{\sigma}_*(\alpha_i)\cdot\widehat{\sigma}_*(\beta_j)\ =\ \widehat{\sigma}_*(\alpha_i)\cdot\beta_j$$

for all j. By elementary properties of the symplectic pairing it now follows that there are elements $a_{ij}\ \in\ \mathbb Z/2\mathbb Z$ such that

$$\widehat{\sigma}_*(\alpha_i)\ =\ \alpha_i+\sum_{j=1}^{g}a_{ij}\beta_j$$

for all $1\ \leq\ i\ \leq\ g$.

Using eqn. (2.7), it follows by induction that

$$\widehat{\sigma}^n_*(\alpha_i)\ =\ \alpha_i+\sum_{j=1}^{g}na_{ij}\beta_j \tag{2.8}$$

for all $n\ \in\ \mathbb N$

As $\widehat{\sigma}$ is a holomorphic automorphism of $X_{\mathbb C}$, and $g\ \geq\ 2$, we know that $\widehat{\sigma}$ is periodic, i.e., $\widehat{\sigma}^n\ =\ \mathrm{Id}_{X_{\mathbb C}}$ for some $n\ >\ 1$. Using eqn. (2.8), this implies that $a_{ij}\ =\ 0$ for all $i\,,j\ \in\ [1\,,g]$. Thus, $\widehat{\sigma}_*$ is the identity automorphism of $H_1(X_{\mathbb C},\,\mathbb Z/2\mathbb Z)$.

Since $\widehat{\sigma}_*$ is the identity automorphism, and $\widehat{\sigma}$ is a nontrivial automorphism of $X_{\mathbb C}$, from [3, Theorem 2.1] it follows that $X_{\mathbb C}$ is a hyperelliptic Riemann surface, and $\widehat{\sigma}_*$ is the unique hyperelliptic involution of $X_{\mathbb C}$.

Therefore, σ is a hyperelliptic involution of X. Thus, if some nontrivial automorphism σ of X acts trivially on $\mathcal S(X)$, then X is hyperelliptic with σ being the unique hyperelliptic involution of X.

To prove the converse it suffices to note that the hyperelliptic involution of a compact hyperelliptic Riemann surface Z acts trivially on the set of all theta characteristics on Z; see [1, p. 288, 32(i)]. This completes the proof of the theorem. ☐

3. AN EXAMPLE

In this section we will construct an example to show that the condition in Theorem 2.3, which says that X does not have any real points, is necessary.

We shall do this by constructing a compact connected Riemann surface F with two commuting antiholomorphic involutions a and b, so that b acts trivially on $H_1(F,\,\mathbb Z/2\mathbb Z)$.

So $h := a \circ b$ and a have the same action in $H_1(F, \mathbb{Z}/2\mathbb{Z})$. In our example, h will be a nontrivial automorphism, of the real algebraic curve X defined by the pair (F, a), satisfying the condition $\text{genus}(F/\langle h\rangle) > 0$. More precisely, h will be a nontrivial automorphism of X that fixes pointwise all the real theta characteristics on X. Since $\text{genus}(F/\langle h\rangle) > 0$, the automorphism h is not a hyperelliptic involution.

The compact Riemann surface F will be constructed using hyperbolic geometry. Note that there is a bijective correspondence between hyperbolic surfaces and Riemann surfaces, with the hyperbolic isometries corresponding to the holomorphic and antiholomorphic self–maps.

Recall that there is a regular right–angled hexagon in the hyperbolic plane $\mathbb{H}$, which is unique up to an isometry of $\mathbb{H}$. Parametrize the sides of this hexagon by $\mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$ preserving their circular orderings.

Let A_1, B_1, A_2 and B_2 be four copies of this hexagon. For each $i \in \{1, 2\}$, and

$$n \in \{0, 2, 4\} \subset \{0, 1, 2, 3, 4, 5\},$$

identify the n–th side of A_i with the n–th side of B_i using the identity map. So, for $i \in \{1, 2\}$, by these identifications we obtain a pair of pants P_i with geodesic boundaries.

Next, for each $i = 1, 2$, identify the 1–st side of $A_i \subset P_i$ with the 3–rd side of $A_i \subset P_i$ using the unique orientation reversing isometry between the two sides. Similarly, identify the 1–st side of $B_i \subset P_i$ with the 3–rd side of $B_i \subset P_i$ using the unique orientation reversing isometry between the two sides. This gives a torus T_i with one boundary component. The boundary of T_i is the circle constructed from the 5–th sides of A_i and B_i.

Let $c_i \subset T_i$ be the geodesic circle which is the image of the union of the 1–st side in A_i and B_i. Let d_i be the simple closed curve in T_i which is a the image of the 2–nd side of A_i; note that in P_i, the 2–nd side of A_i is identified with the 2–nd side of B_i. Finally, let a_i (respectively, b_i) be the image in T_i of the 5–th side of A_i (respectively, B_i). Therefore, $a_i = A_i \bigcap \partial T_i$ and $b_i = B_i \bigcap \partial T_i$.

We now construct F from T_1 and T_2 by identifying their boundary components in such a manner that the arcs a_1 and a_2 are identified, while b_1 is identified with b_2. More precisely, Consider the unique orientation reversing isometry from a_1 (respectively, b_1) to a_2 (respectively, b_2). Let F be the quotient of the disjoint union of T_1 and T_2 constructed using these two isometries. Using the restriction of the hyperbolic metric on $\mathbb{H}$ to the regular hexagon, we obtain a hyperbolic metric on F. Therefore, F is equipped with a complex structure.

Consider the unique orientation reversing isometries

$$A_1 \longrightarrow A_2 \quad \text{and} \quad B_1 \longrightarrow B_2$$

that map a_1 to a_2 and b_1 to b_2 respectively. These two isometries and their inverses together induce an orientation reversing isometry

$$a\,:\,F\,\longrightarrow\,F \tag{3.1}$$

of the quotient F of the disjoint union of A_1, A_2, B_1 and B_2.

Similarly, we have an orientation reversing isometry

$$A_1\,\longrightarrow\,B_1$$

(respectively, $A_2\,\longrightarrow\,B_2$) that takes a_1 to b_1 (respectively, a_2 to b_2). These, together with their inverses, induce an orientation reversing isometry

$$b\,:\,F\,\longrightarrow\,F \tag{3.2}$$

of the quotient F. Also, define

$$h\,:=\,a\circ b\,. \tag{3.3}$$

Therefore, h is an orientation preserving isometry of F. Hence, h is a holomorphic automorphism of the Riemann surface F. We note that both a and b are involutions.

The isometry b in eqn. (3.2) fixes the curves $c_1\,,c_2\,,d_1\,,d_2$ as sets (though not pointwise). As these generate $H_1(F,\,\mathbb{Z}/2\mathbb{Z})$, it follows that b acts trivially on $H_1(F,\,\mathbb{Z}/2\mathbb{Z})$. Therefore, the diffeomorphism h in eqn. (3.3) fixes all the elements of $H_1(F,\,\mathbb{Z}/2\mathbb{Z})$ that are fixed by a. From this it follows immediately that h fixes all the elements of $H^1(F,\,\mathbb{Z}/2\mathbb{Z})$ that are fixed by a.

The pair $(F\,,a)$ define a geometrically irreducible smooth projective curve of genus two defined over $\mathbb{R}$. Let X denote the real curve defined by $(F\,,a)$. We note that the image in F of the 5–th side of A_1 is fixed pointwise by a. Therefore, the real curve X has real points.

Clearly we have $a\circ b\,=\,b\circ a$, where a and b are constructed in eqn. (3.1) and eqn. (3.2) respectively. Therefore, h defines an automorphism of X. Let

$$h'\,:\,X\,\longrightarrow\,X \tag{3.4}$$

be the automorphism of the real algebraic curve X given by h. Since both a and b are involutions, it follows that h' is also an involution.

We will show that the above automorphism h' fixes pointwise all the real theta characteristics on X.

We have already seen that h fixes all the elements of $H^1(F,\,\mathbb{Z}/2\mathbb{Z})$ that are fixed by a. Since the homomorphism ϕ in eqn. (2.5) commutes with the action of h', to prove that $h'\,\in\,\mathrm{Aut}(X)$ fixes pointwise all the real theta characteristics on X it suffices to show that there is one real theta characteristic on X which is fixed by h'.

To construct a real theta characteristic on X which is fixed by h', let

$$\{p^0_i\,,p^1_i\}\,\subset\,A_i\,\subset\,F$$

be the two vertices of the 1–st side of the hexagon A_i, where $i = 1, 2$. The mid–point of the 2–nd side of A_i will be denoted by q_i. The two vertices of the 5–th side of $A_1 \subset F$ will be denoted by x^0 and x^1. Note the images in F of the two vertices of the 5–th side of A_2, B_1 and B_2 all coincide with the subset $\{x^0, x^1\} \subset F$.

Lemma 3.1. *Let $\mathcal{O}_F(D)$ be the holomorphic line bundle on F given by the divisor*

$$D := p_1^0 + p_1^1 + q_1 - x^0 - x^1 .$$

Then $\mathcal{O}_F(D)$ is a real theta characteristic on the real curve $X := (F, a)$. The theta characteristic $\mathcal{O}_F(D)$ is fixed by the automorphism h' of X defined in eqn. (3.4).

Proof. We need an explicit construction of the hyperelliptic involution of F. For this, consider the reflections on the hexagons A_i and B_i about the geodesic line joining the midpoints of the 2–nd and 5–th sides. These combine together to induce an orientation reversing isometry

$$c : F \longrightarrow F .$$

Clearly, c is an involution. Define

$$f := c \circ b ,$$

where b is defined in eqn. (3.2). Therefore, f is an orientation preserving isometry of F. Hence f is a holomorphic automorphism of the Riemann surface F.

It is easy to see that the two automorphisms c and b commute. Since both b and c are involutions, it follows that f is also an involution. It is easy to check that

$$\text{genus}(F/\langle f\rangle) = 0 .$$

Hence f is the unique hyperelliptic involution of F.

The fixed points of the hyperelliptic involution of F are precisely the Weierstrass points of F. Hence the fixed–point set

$$F^f := \{p_1^0, p_1^1, q_1, p_2^0, p_2^1, q_2\} \subset F$$

is the set of Weierstrass points of F, where p_i^0, p_i^1 and q_i are as in the statement of the lemma.

Also note the subset $\{x^0, x^1\} \subset F$ is left invariant by the involution of f. As a consequence, the holomorphic line bundle on F given by the divisor $x^0 + x^1$ is isomorphic to the pull back $\widehat{f}^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$, where

$$\widehat{f} : F \longrightarrow F/\langle f\rangle \cong \mathbb{P}^1_{\mathbb{C}}$$

is the quotient map, and $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ is the unique line bundle of degree one over $\mathbb{P}^1_{\mathbb{C}}$.

Let $D' := p_2^0 + p_2^1 + q_2 - x^0 - x^1$ be the divisor on F. Let $\mathcal{O}_F(D')$ be the holomorphic line bundle over F defined by D'.

Using the above descriptions of $\widehat{f}^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ and the Weierstrass points of F, from the description, given in [1, p. 288, 32(i)], of the theta characteristics on F it follows immediately that both $\mathcal{O}_F(D)$ and $\mathcal{O}_F(D')$ are theta characteristics on F; the line bundle

$\mathcal{O}_F(D)$ is defined in the statement of the lemma. Furthermore, from [1, p. 288, 32(2)] it follows immediately that $\mathcal{O}_F(D)$ and $\mathcal{O}_F(D')$ give the same theta characteristics on F, or in other words, the holomorphic line bundle $\mathcal{O}_F(D')$ is isomorphic to $\mathcal{O}_F(D)$.

The map a in eqn. (3.1) takes the divisor D to the divisor D'. Hence $\mathcal{O}_F(D)$ is a real theta characteristic on the real curve $X := (F\,,a)$. The holomorphic isomorphism h in eqn. (3.3) takes the divisor D to the divisor D'. Consequently, the isomorphism h' of X constructed in eqn. (3.4) fixes the real theta characteristic defined by $\mathcal{O}_F(D)$. This completes the proof of the lemma. $\square$

As we noted prior to Lemma 3.1, from Lemma 3.1 it follows that h' fixes pointwise all the real theta characteristics on X. On the other hand, we have

$$\text{genus}(F/\langle h\rangle) \,=\, 1\,.$$

Hence h' is not the hyperelliptic involution of X. We also noted earlier that X has real points. Therefore, the automorphism h' of the real curve X is an example showing that the condition in Theorem 2.3 that X does not have any real points is necessary.

It is easy to check that the holomorphic line bundle $\mathcal{O}_F(D)$ over F in Lemma 3.1 defines a real algebraic line bundle over the real curve $X := (F\,,a)$.

Taking the unique regular right–angled $2(2m+1)$–gon in the hyperbolic plane in place of the regular hexagon, higher genus examples can be constructed by a similar method.

4. Odd theta characteristics on a Riemann surface

We start with a topological lemma.

Let F be a compact connected Riemann surface with a holomorphic involution

$$\gamma\,:\,F\,\longrightarrow\,F$$

such that the quotient $F/\langle\gamma\rangle$ is the projective line $\mathbb{P}^1_{\mathbb{C}}$. Let

$$f\,:\,F\,\longrightarrow\,\mathbb{P}^1_{\mathbb{C}}$$

be the quotient map. Let $\mathcal{R}\,\subset\,\mathbb{P}^1_{\mathbb{C}}$ be the subset over which f is ramified.

For any pair of ramification points $\{x\,,y\}\,\subset\,\mathcal{R}$, let $C(x,y)$ be a simple closed curve that separates this pair from the other ramification points. Then $f^{-1}(C(x,y))$ is a pair of simple closed curves. We shall see that these two curves represent the same element in $H_1(Y,\,\mathbb{Z}/2\mathbb{Z})$. Let $C'(x,y)$ be one of these two curves.

Lemma 4.1. *The collection of curves $C'(x,y)$, where $\{x\,,y\}$ run over all subsets of $\mathcal{R}$ of cardinality two, generate $H_1(F,\,\mathbb{Z}/2\mathbb{Z})$.*

Proof. Let $2k$ be the cardinality of $\mathcal{R}$. We will prove the lemma by induction on k. For $k\,=\,0\,,1$, the surface F is a sphere and hence $H_1(F,\,\mathbb{Z}/2\mathbb{Z})$ is trivial. Therefore, the lemma is proved for $k\,=\,0\,,1$.

Now assume that the lemma holds for all $k \in [0\,, k_0]$, where $k_0 \geq 1$ is some integer. Take a pair $(F'\,, f')$, where $\text{genus}(F') = k_0 - 1$ and

$$f \,:\, F' \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$

is a double cover. Pick a disk D in $\mathbb{P}^1_{\mathbb{C}}$ disjoint from the $2k_0$ ramification points of f'. The inverse image of D in F' is a pair of disks D_1 and D_2. As the ramified cover of a disk with two ramification points is an annulus, it follows that surface F, for $k = k_0+1$, is obtained from F' by removing the interiors of the disks D_1 and D_2 and attaching an annulus A. Furthermore, we can take $C(p, q)$ to be the boundary of the disk D. It follows that the inverse image of $C(p, q)$ consists of the boundary components of the annulus A, which are homologous curves in $H_1(F,\, \mathbb{Z}/2\mathbb{Z})$. Denote the class represented by these curves as β.

By the above, F is obtained from F' by adding a 1-handle. Hence the rank of $H_1(F,\, \mathbb{Z}/2\mathbb{Z})$ is two more that that of $H_1(F',\, \mathbb{Z}/2\mathbb{Z})$, with β independent of $H_1(F',\, \mathbb{Z}/2\mathbb{Z})$. Further, assume that D is chosen in such a manner that the generators of F' given inductively are disjoint from D_1 and D_2. Then β is disjoint from all the given generators of $H_1(F',\, \mathbb{Z}/2\mathbb{Z})$. Hence if $\delta \in H_1(F',\, \mathbb{Z}/2\mathbb{Z})$, then $\beta \cdot \delta = 0$. (As in Section 2, for any $v\,, w \in H_1(F',\, \mathbb{Z}/2\mathbb{Z})$, by $v \cdot w$ we will denote the element in $\mathbb{Z}/2\mathbb{Z}$ given by the cap product of them.) Since the intersection pairing on the homology of a surface is a symplectic pairing, it follows that if $\alpha \in H_1(F,\, \mathbb{Z}/2\mathbb{Z})$ is an element with $\alpha \cdot \beta = 1$, then the generators of $H_1(F',\, \mathbb{Z}/2\mathbb{Z})$ together with α and β generate $H_1(F,\, \mathbb{Z}/2\mathbb{Z})$

Now let r be a ramification point different from p and q. We can choose $C(p, r)$ to intersect $C(p, q)$ transversally in two points x and y. Observe that there is a curve consisting of arcs in $C(p, q)$ and $C(p, r)$ joining x and y that separates p from the other ramification points, and hence does not lift to F. It follows that $C'(p, r)$ and $C'(p, q)$ intersect transversally in a single point. Hence, if α is the element represented by $C'(p, r)$, then $\alpha \cdot \beta = 1$.

It follows from the above that the generators of $H_1(F',\, \mathbb{Z}/2\mathbb{Z})$ together with the elements α and β represented by $C'(p, r)$ and $C'(p, q)$ respectively generate $H_1(F,\, \mathbb{Z}/2\mathbb{Z})$. The lemma follows by induction. □

Let Y be a compact connected Riemann surface. A holomorphic line bundle L over Y is called an *odd theta characteristic* if

- $L^{\otimes 2}$ is holomorphically isomorphic to the holomorphic cotangent bundle K_Y, and
- $\dim H^0(Y,\, L)$ is an odd integer.

The set of all odd theta characteristics on Y will be denoted by $\mathcal{S}_1(Y)$. It is known that if $g := \text{genus}(Y) > 1$, then the cardinality of $\mathcal{S}_1(Y)$ is $2^{g-1}(2^g - 1)$ [5, p. 190, § 4].

Any holomorphic automorphism of Y clearly acts on $\mathcal{S}_1(Y)$. Our aim in this section is to prove the following proposition:

Proposition 4.2. *Let Y be a compact connected Riemann surface of genus at least two that admits a nontrivial holomorphic automorphism. Fix a nontrivial holomorphic automorphism σ of Y. If σ fixes pointwise all the elements of $\mathcal{S}_1(Y)$, then Y is hyperelliptic with σ being the unique hyperelliptic involution of Y.*

Proof. Let g be the genus of Y; so $g \geq 2$. Let $\mathcal{S}(Y)$ denote the set of all theta characteristics on Y. The set $\mathcal{S}(Y)$ is principal homogeneous space for $H^1(Y, \mathbb{Z}/2\mathbb{Z}) = H_1(Y, \mathbb{Z}/2\mathbb{Z})$ (the isomorphism is given by the Poincaré duality pairing). Let

$$\mathcal{T}(Y) \subset H_1(Y, \mathbb{Z}/2\mathbb{Z}) \tag{4.1}$$

be the linear subspace generated by all elements $c \in H_1(Y, \mathbb{Z}/2\mathbb{Z})$ such that there exist

$$\alpha\,,\beta \in \mathcal{S}_1(Y)$$

with $\alpha = \beta + c$. To prove the proposition it suffices to show that $\mathcal{T}(Y) = H_1(Y, \mathbb{Z}/2\mathbb{Z})$. Indeed, in that case the proposition follows from [3, Theorem 2.1].

In a holomorphic family of compact connected Riemann surfaces with theta characteristics, parametrized by a connected space, the parity of the theta characteristics remain fixed [2, p. 48, Theorem 1], [5, p. 184, Theorem]. Since the moduli space of Riemann surfaces of genus g is connected, this implies that the integer $\dim \mathcal{T}(Y)$ depends only on g; in particular, $\dim \mathcal{T}(Y)$ is independent of the complex structure of Y. Therefore, to prove that $\mathcal{T}(Y) = H_1(Y, \mathbb{Z}/2\mathbb{Z})$, we may assume that Y is a hyperelliptic Riemann surface.

Assume that Y is hyperelliptic. Let

$$\gamma : Y \longrightarrow Y$$

be the hyperelliptic involution. Let

$$f : Y \longrightarrow Y/\langle\gamma\rangle = \mathbb{P}^1_{\mathbb{C}}$$

be the quotient map. So f is ramified over $2g+2$ points. Let $\mathcal{R} \subset \mathbb{P}^1_{\mathbb{C}}$ be the subset over which f is ramified.

For any pair of ramification points $\{x\,,y\} \subset \mathcal{R}$, let $C'(x,y)$ be the closed curve in Y considered in Lemma 4.1.

Setting $m = 0$ in [1, p. 288, 32(i)], from [1, p. 288, 33] we conclude that the element in $H_1(Y, \mathbb{Z}/2\mathbb{Z})$ defined by $C'(x,y)$ lies in $\mathcal{T}(Y)$ for all $\{x\,,y\} \subset \mathcal{R}$. Therefore, the proposition follows from the Lemma 4.1. ☐

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