

# ON THE TOPOLOGY OF MANIFOLDS WITH POSITIVE ISOTROPIC CURVATURE

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ABSTRACT. We show that a closed orientable Riemannian  $n$ -manifold,  $n \geq 5$ , with positive isotropic curvature and free fundamental group is homeomorphic to the connected sum of copies of  $S^{n-1} \times S^1$ .

## 1. INTRODUCTION

Let  $(M, g)$  be a closed, orientable, Riemannian manifold with positive isotropic curvature. By [9], if  $M$  is simply-connected then  $M$  is homeomorphic to a sphere of the same dimension. We shall generalise this to the case when the fundamental group of  $M$  is a free group.

**Theorem 1.1.** *Let  $M$  be a closed, orientable Riemannian  $n$ -manifold with positive isotropic curvature. Suppose that  $\pi_1(M)$  is a free group on  $k$  generators. Then, if  $n \neq 4$  or  $k = 1$  (i.e.  $\pi_1(M) = \mathbb{Z}$ ),  $M$  is homeomorphic to the connected sum of  $k$  copies of  $S^{n-1} \times S^1$ .*

We note that a conjecture of M. Gromov( [4] Section 3 (b)) and A. Fraser [2], based on the work of Micallef-Wang [8], states that any compact manifold with positive isotropic curvature has a finite cover satisfying our hypothesis.

**Conjecture 1** (M. Gromov-A. Fraser).  *$\pi_1(M)$  is virtually free, i.e., it is a finite extension of a free group.*

It is known by the work of A. Fraser [2] and A. Fraser and J. Wolfson [3] that  $\pi_1(M)$  does not contain any subgroup isomorphic to the fundamental group of a closed surface of genus at least one.

Our starting point is the following fundamental result of M. Micallef and J. Moore [9].

**Theorem 1.2** (M. Micallef-J. Moore). *Suppose  $M$  is a closed manifold with positive isotropic curvature. Then  $\pi_i(M) = 0$  for  $2 \leq i \leq \frac{n}{2}$ .*

It is clear that the following purely topological result, together with the Micallef-Moore theorem, implies Theorem 1.1

**Theorem 1.3.** *Let  $M$  be a smooth, orientable, closed  $n$ -manifold such that  $\pi_1(M)$  is a free group on  $k$  generators and  $\pi_i(M) = 0$  for  $2 \leq i \leq \frac{n}{2}$ . If  $n \neq 4$  or  $k = 1$ , then  $M$  is homeomorphic to the connected sum of  $k$  copies of  $S^{n-1} \times S^1$ .*

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1991 *Mathematics Subject Classification.* Mathematics Subject Classification (1991): Primary 53C21, Secondary 53C20.

Henceforth let  $M$  be a smooth, orientable, closed  $n$ -manifold such that  $\pi_1(M)$  is a free group on  $k$  generators and  $\pi_i(M) = 0$  for  $2 \leq i \leq \frac{n}{2}$ . We assume throughout that all manifolds we consider are orientable.

Let  $\widetilde{M}$  be the universal cover of  $M$ . Hence  $\pi_1(\widetilde{M})$  is trivial and so is  $\pi_i(\widetilde{M}) = \pi_i(M)$  for  $2 \leq i \leq \frac{n}{2}$ . We shall show that the homology of  $\widetilde{M}$  is isomorphic as  $\pi_1(M)$ -modules to that of the connected sum of  $k$  copies of  $S^{n-1} \times S^1$ . We then show that  $M$  is homotopy equivalent to the connected sum of  $k$  copies of  $S^{n-1} \times S^1$  using Theorems of Whitehead. Finally, recent results of Kreck and Lück allow us to conclude the result.

*Acknowledgements.* We thank the referees for helpful comments and for suggesting a simplification of our proof.

## 2. THE HOMOLOGY OF $\widetilde{M}$

Let  $X$  denote the wedge  $\bigvee_{j=1}^k S^1$  of  $k$  circles and let  $x$  denote the common point on the circles. Choose and fix an isomorphism  $\varphi$  from  $\pi_1(M, p)$  to  $\pi_1(X, x)$  for some basepoint  $p \in M$ . We shall use this identification throughout. Denote  $\pi_1(M, p) = \pi_1(X, x)$  by  $\pi$ .

As  $X$  is an Eilenberg-MacLane space, there is a map  $f : (M, p) \rightarrow (X, z)$  inducing  $\varphi$  on fundamental groups and a map  $s : (X, z) \rightarrow (M, p)$  so that  $f \circ s : X \rightarrow X$  is homotopic to the identity.

We deduce the homology of  $\widetilde{M}$  using the Hurewicz Theorem and Poincaré duality.

**Lemma 2.1.** *For  $1 \leq i \leq n/2$ ,  $H_i(\widetilde{M}, \mathbb{Z}) = 0$*

*Proof.* As  $\widetilde{M}$  is simply-connected and  $\pi_i(\widetilde{M}) = \pi_i(M) = 0$  for  $1 < i \leq n/2$  (by hypothesis), by Hurewicz theorem  $H_i(\widetilde{M}, \mathbb{Z}) = 0$  for  $1 \leq i \leq n/2$ .  $\square$

We deduce the homology in dimensions above  $n/2$  using Poincaré duality for  $M$  with coefficients in the module  $\mathbb{Z}[\pi]$ , namely

$$H_{n-i}(M, \mathbb{Z}[\pi]) = H^i(M, \mathbb{Z}[\pi])$$

Recall that  $H_k(M, \mathbb{Z}[\pi]) = H_k(\widetilde{M}, \mathbb{Z})$  and the group  $H^i(M, \mathbb{Z}[\pi])$  is the cohomology with compact support  $H_c^i(M, \mathbb{Z})$ . Hence Poincaré duality with coefficients in  $\mathbb{Z}[\pi]$  is the same as Poincaré duality for a non-compact manifold relating homology to cohomology with compact support.

To apply Poincaré duality, we need the following lemma.

**Lemma 2.2.** *For  $1 \leq i \leq n/2$ , the map  $s : (X, z) \rightarrow (M, p)$  induces isomorphisms of modules with  $s_* : H^i(M; \mathbb{Z}[\pi]) \rightarrow H^i(X; \mathbb{Z}[\pi])$ .*

*Proof.* As the map  $s$  induces an isomorphism on homotopy groups in dimensions at most  $n/2$ , it induces isomorphisms on the cohomology groups with twisted coefficients. Specifically, we can add cells of dimensions  $k \geq n/2 + 2$  to  $M$  to obtain an Eilenberg-MacLane space  $\bar{M}$  for the group  $\pi$ , which is thus homotopy equivalent to  $X$ . For  $i \leq n/2$  and any  $\mathbb{Z}[\pi]$ -module  $A$ , it follows that

$$H_i(M, A) = H_i(\bar{M}, A) = H_i(X, A)$$

where the first equality follows as the cells added to  $M$  to obtain  $\bar{M}$  are of dimension at least  $n/2 + 2$  and the second as the spaces are homotopy equivalent.  $\square$

By applying Poincaré duality, we obtain the following result.

**Lemma 2.3.** *Let  $M$  be a smooth, orientable, closed  $n$ -manifold such that  $\pi_1(M)$  is a free group on  $k$  generators and  $\pi_i(M) = 0$  for  $2 \leq i \leq \frac{n}{2}$ . Then, for the universal cover  $\widetilde{M}$  of  $M$ ,*

- (1)  $H_i(\widetilde{M}, \mathbb{Z}) = 0$  for  $1 \leq i < n - 1$
- (2) *We have an isomorphism  $H_{n-1}(\widetilde{M}, \mathbb{Z}) = H_c^1(\widetilde{X}, \mathbb{Z})$ , where  $\widetilde{X}$  is the universal cover of  $X$ , determined by the isomorphisms  $s_* : \pi_1(X, z) \rightarrow \pi_1(M, p)$  on fundamental groups.*

*Proof.* The statements follow from Lemmas 2.1 and 2.2 by using  $H_*(\widetilde{M}, \mathbb{Z}) = H_*(M, \mathbb{Z}[\pi])$ .  $\square$

### 3. HOMOTOPY TYPE

We now show that  $M$  is homotopy equivalent to the connected sum  $Y$  of  $k$  copies of  $S^{n-1} \times S^1$ . Our first step is to construct a map  $g : Y \rightarrow M$ . We shall then show that it is a homotopy equivalence.

Note that  $Y$  has the structure of a CW-complex obtained as follows. The 1-skeleton of  $Y$  is the wedge  $X$  of  $k$  circles. Let  $\alpha_i$  denote the  $i$ th circle with a fixed orientation.

We attach  $k$   $(n - 1)$ -cells  $D_j$ , with the  $j$ th attaching map mapping  $\partial D^{n-1}$  to the midpoint  $x_j$  of the  $j$ th circle. Finally, we attach a single  $n$ -cell  $\Delta$ .

We associate to  $D_j$  an element  $A_j \in \pi_{n-1}(Y, x)$ . Namely, as the attaching map is constant, the  $j$ th  $(n - 1)$ -cell gives an element  $B_j \in \pi_{n-1}(Y, x_j)$ . We consider the subarc  $\beta_j$  of  $\alpha_j$  joining  $z_j$  to  $x$  in the negative direction and let  $A_j$  be obtained from  $B_j$  by the change of basepoint isomorphism using  $\beta_j$ .

Note that if we instead chose the arc joining  $z_j$  to  $x$  in the positive direction, then the resulting element is  $-\alpha_j \cdot A_j$ . By the construction of  $Y$ , it follows that the attaching map of the  $(n - 1)$ -cell represents the element

$$\partial\Delta = \Sigma_j(A_j - \alpha_j \cdot A_j)$$

in  $\pi_{n-1}(Y)$  regarded as a module over  $\pi_1(Y)$ . This can be seen for instance by using Poincaré duality.

We now construct the map  $g : Y \rightarrow M$ . Recall that we have a map  $s : (X, z) \rightarrow (M, p)$  inducing the isomorphism  $\varphi^{-1}$  on fundamental groups. We define  $g$  on the 1-skeleton  $X$  of  $Y$  by  $g|_X = s$ . We henceforth identify the fundamental groups of  $Y$  and  $M$  using the isomorphism  $\varphi$ , i.e.,  $\pi_1(Y, z)$  is identified with  $\pi$ .

We next extend  $g$  to the  $n$ -cell of  $Y$  as follows. By Hurewicz theorem and Lemma 2.3, we have isomorphisms of  $\pi$ -modules  $\pi_{n-1}(M, p) = H_{n-1}(\widetilde{M}, \mathbb{Z})$  and  $\pi_{n-1}(Y, z) = H_{n-1}(\widetilde{M}, \mathbb{Z})$ . By Lemma 2.3, each of these modules is isomorphic to  $H_c^1(\widetilde{X}, \mathbb{Z})$  with the isomorphisms determined by the identifications of the fundamental groups.

Under the above isomorphisms the elements  $A_j$  correspond to elements  $A'_j$  in  $\pi_{n-1}(M, p)$ . Consider the element  $B'_j$  of  $\pi_{n-1}(M, g(z_j))$  obtained from  $A'_j$  by the basechange map using the arc  $f(\beta_j)$ . We define the map  $g$  on  $D_j$  extending the constant map on its boundary to be a representative of  $B'_j$ .

As the  $\pi$ -modules  $\pi_{n-1}(M, p)$  and  $\pi_{n-1}(Y, z)$  are isomorphic, the image  $g$  of  $\partial\Delta$  is homotopically trivial. Hence we can extend the map  $g$  across the cell  $\Delta$ .

**Lemma 3.1.** *The map  $g : Y \rightarrow M$  is a homotopy equivalence.*

*Proof.* Let  $G : \tilde{Y} \rightarrow \tilde{M}$  be the induced map on the universal covers. By Lemma 2.3 applied to  $M$  and  $Y$ , we see that  $H_p(\tilde{Y}) = H_p(\tilde{M}) = 0$  for  $0 < p \neq n - 1$  and  $G$  induces an isomorphism on  $H_{n-1}$ . Thus the map  $G$  is a homology equivalence. By a theorem of Whitehead [10], a homology equivalence between simply-connected CW-complexes is a homotopy equivalence.

It follows that  $G$  induces isomorphisms  $G_* : \pi_k(\tilde{Y}) \rightarrow \pi_k(\tilde{M})$  for  $k > 1$ . As covering maps induce isomorphisms on higher homotopy groups, and  $g$  induces an isomorphism on  $\pi_1$ , it follows that  $g$  is a weak homotopy equivalence, hence a homotopy equivalence (see [5]).  $\square$

#### 4. PROOF OF THEOREM 1.3

The rest of the proof of Theorem 1.3 is based on results of Kreck-Lück [7]. In [7], the authors define a manifold  $N$  to be a *Borel manifold* if any manifold homotopy equivalent to  $N$  is homeomorphic to  $N$ . We have shown that a manifold  $M$  satisfying the hypothesis of Theorem 1.3 is homotopy equivalent to the connected sum  $Y$  of  $k$  copies of  $S^{n-1} \times S^1$ . Hence it suffices to observe that  $Y$  is Borel.

By Theorem 0.13(b) of [7], the manifold  $S^{n-1} \times S^1$  is Borel for  $n \geq 4$ . This completes the proof in the case when  $\pi_1(M) = \mathbb{Z}$ . Further, if  $n \geq 5$ , then Theorem 0.9 of [7] says that the connected sum of Borel manifolds is Borel, hence  $Y$  is Borel. This concludes the proof for  $\pi_1(M)$  a free group and  $n \geq 5$ .  $\square$

Finally, in the case when  $n = 3$  by the Kneser conjecture (proved by Stallings) the manifold  $M$  is a connected sum of manifolds whose fundamental group is  $\mathbb{Z}$ . As  $M$  is orientable, it follows that if  $M$  is expressed as a connected sum of prime manifolds (such a decomposition exists and is unique by the Kneser-Milnor theorem), then each prime component is either  $S^2 \times S^1$  or a homotopy sphere. By the Poincaré conjecture (Perelman's theorem), every homotopy 3-sphere is homeomorphic to a sphere. It follows that  $M$  is the connected sum of  $k$  copies of  $S^2 \times S^1$ .  $\square$

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