DEGREE-ONE MAPS AND AKBULUT CORKS

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ABSTRACT. Suppose that M and N are smooth, simply-connected, closed 4-manifolds and $f : M \to N$ is a degree-one map. We show that there is a map $g : M \to N$ homotopic to f and a compact, contractible, smooth 4-manifold $C \subset N$ such that $g^{-1}(C)$ is simply-connected and the restriction of g from the complement of $g^{-1}(C)$ to the complement of C is a diffeomorphism if a certain associated form Q is the intersection form of a closed, smooth, oriented manifold with a handle-decomposition without 3-handles. In particular the condition depends only on the homotopy types of M and N. This generalises the result that if M and N are homotopy equivalent, then M can be obtained from N by replacing a contractible manifold in N by a different contractible manifold.

1. Introduction

Let M and N be smooth, simply-connected, closed 4-manifolds. Suppose that M and N are homotopy equivalent, then they are homeomorphic by Freedman's theorem but need not be diffeomorpic. However, it is known that they are diffeomorphic outside contractible sets, i.e., there are compact, contractible 4-manifolds contained in M and N so that the complement of these manifolds are diffeomorphic(see [3], [5] and [6]). Note that conversely manifolds that differ outside compact sets are homotopy equivalent.

In this note, we generalise this result to degree-one maps $f : M \to N$. Observe that if f is not a homotopy equivalence, then f cannot be a diffeomorphism outside compact, contractible subsets of both M and N. However, we show that, up to homotopy, the map is a diffeomorphism from the complement of a compact, simply-connected 4-manifold in M to the complement of a compact, contractible 4-manifold of N provided an associatd quadratic form is representible by a smooth, simply-connected, closed 4-manifold without 3-handles.

Before stating our main result we need some preliminaries regarding degree-one maps. We begin with some basic result of Hopf concerning degree-one maps.

Lemma 1.1 (Hopf)**.** *The map $f^* : H^(N) \to H^2(M)$ preserves the intersection pairing.*

Proof. As f has degree one, $f_*([M]) = [N]$. Hence for classes α and β in $H^2(N)$,

$$(\alpha \cup \beta)[N] = (\alpha \cup \beta) f_*([M]) = (f^*(\alpha) \cup f^*(\beta))[M]$$

□

Lemma 1.2 (Hopf)**.** *Let $f : M \to N$ be a degree-one map as above. Then $f^* : H^2(N) \to H^2(M)$ is an injection and its image is a direct summand.*

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Proof. If $\alpha \in H^2(N)$ is non-zero, then by Poincaré duality there is a class $\beta \in H^2(N)$ with $(\alpha \cup \beta)[N] \neq 0$. It follows that $(f^*(\alpha) \cup f^*(\beta))[M] \neq 0$ and hence $f^*(M) \neq 0$. Thus f^* is injective and its image is a subgroup of $H^2(M)$.

Thus, to prove that the image is a direct summand, it suffices to show that the image of a primitive element is primitive. If $\alpha \in H^2(N)$ is primitive, then by Poincaré duality there is a class $\beta \in H^2(N)$ with $(\alpha \cup \beta)[N] = 1$. It follows that $(f^*(\alpha) \cup f^*(\beta))[M] = 1$ and hence $f^*(M)$ is primitive.

□

We identify the image of $H^2(N)$ with $H^2(N)$. Note that this respects the bilinear form induced by the cup product. Thus we have a decomposition of symmetric bilinear forms.

$$H^2(M) = H^2(N) \oplus Q$$

It follows that the form Q is symmetric and unimodular.

We can now state our main result. Let M and N are smooth, simply-connected, closed 4-manifolds and $f : M \to N$ is a degree-one map. Let Q be the bilinear form constructed above.

Theorem 1.3. *There is a map $g : M \to N$ homotopic to f and a smooth, compact, contractible 4-manifold $C \subset N$ such that $g^{-1}(C)$ is simply-connected and the restriction of g from the complement of $g^{-1}(C)$ to the complement of C is a diffeomorphism if $Q = H^2(X)$ for a simply-connected, closed, oriented, smooth manifold X with a handle-decomposition without 3-handles.*

We note that any map $g : M \to N$ between smooth 4-manifolds satisfying the conclusion of the above theorem has degree one. We also have a partial converse to our statement.

Proposition 1.4. *Let $f : M \to N$ be a degree-one map as above. Suppose there is a map $g : M \to N$ and a 4-manifold $C \subset N$ satisfying the conclusions of Theorem 1.3. Then $Q = H^2(X)$ for a simply-connected, smooth, closed, oriented 4-manifold X.*

Proof. Let g and C be as in the hypothesis, and let V and W be the complements of the interiors of C and $g^{-1}(C)$ respectively. Then $\partial V = \partial W$ is a homology sphere, which we denote by N. Let $X = V \coprod_N \bar{W}$ be obtained by attaching V along its boundary N to the manifold $\bar{W}$ that is obtained from W by reversing orientations. Then $Q = H^2(X)$ and X is a simply-connected, smooth, closed, oriented 4-manifold. □

Conjecturally all simply-connected, smooth, closed, 4-manifolds have a handle-decomposition without 3-handles. If this conjecture holds, then our results give a complete characterisation for when there is an Akbulut cork type map homotopic to a degree-one map.

Remark 1.5. In the case of homotopy equivalent smooth four-manifolds, M can be obtained from N by removing a compact, contractible 4-manifold and gluing back the same manifold using a different identification map, which can in fact be taken to be an involution (the so called *Akbulut cork* construction). In our case, as $g^{-1}(C)$ is not even contractable, we clearly cannot have a counterpart of this statement.

We see that (a weak form) of the Akbulut cork theorem follows from our main result.

Corollary 1.6. *Let M and N be homotopy-equivalent closed,, oriented, smooth 4-manifolds. Then there are compact, contractible, smooth 4-manifolds $C \subset M$ and $C' \subset N$ the complements of whose interiors are diffeomorphic.*

Our proof is based on the principle that after taking connected sums with sufficiently many copies of $S^2 \times S^2$ (stabilisation) the homotopy typse of manifolds determine their diffeomorphism types. Further, as in the proof of the 'Akbulut corks' theorem, the surgeries to undo the stabilisation can be arranged to be supported in manifolds that after surgery are simply-connected and either have trivial homology or homology that is in some sense *minimal*.

2. Proof of Theorem 1.3

Let M, N and $f : M \to N$ be as in the hypothesis of Theorem 1.3 and let X be a closed, oriented smooth manifold with $H^2(X) = Q$.

Then M is homotopy equivalent to $N\#X$. By a theorem of Wall [7], it follows that for some integer k there is a diffeomorphism $M\#_k S^2 \times S^2 = N\#X\#_k S^2 \times S^2 = W$.

Note that corresponding to the above expressions of W as connected sums, we have the decompositions

$$H^2(W) = H^2(M) \oplus k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] = H^2(N) \oplus (Q \oplus k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$$

By a theorem of Wall ([7], see Theorem 9.2.13 of [4]), by incresing k if necessary we can choose the diffeomorphism so that for the induced map in homology the components of the form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ in the above decomposition are identified pairwise and so that $H^2(N)$ in the second decomposition is identified with its image in $H^2(M)$ under the map f. It follows that we have a natural inclusion of $H^2(N)$ in $H^2(W)$

We note that M can be obtained from W by performing surgeries about k disjoint spheres $S_1, \dots S_k$. We can obtain N from W by first performing surgery about k disjoint spheres S'_1, S'_2, $\dots$, S'_k to get a manifold containing X as a summand (under connected sum), and then removing X from this. Further, the spheres can be chosen so that S'_i is dual to S_i for $1 \leq i \leq k$, i.e., so that the corresponding bilinear form on their span is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Our main construction, modelled on the Akbulut's cork theorem, is to construct a simply-connected, compact 4-manifold $B \subset W$ containing all the above spheres S_i and S'_j as well as X and so that $H^2(B)$ is complementary to $H^2(N)$ in the above decomposition. Recall that $H^2(N)$ is a direct summand of $H^2(M)$.

Lemma 2.1. *There is a compact simply-connected, smooth 4-manifold $B \subset W$ containig X, the spheres S_i and the spheres S'_j so that $H^2(W) = H^2(N) \oplus H^2(B)$.*

Proof. We begin by constructing a handle-decomposition of W of an appropriate form. To do this, consider first a handle-decomposition of the complement of a ball in X without 3-handles (and with only one 0-handle). We can assume that the 0-handle and the 1-handles are disjoint from the spheres S_i and the spheres S'_j. Let $D_1, \dots, D_m$ be the co-cores of the 2-handles.

We shall modify this handle-decomposition. As each sphere S'_j is dual to S_j, there are points p'_j of intersection between these pairs of spheres. Pick disjoint arcs from the 0-handle to each p'_j, with interiors disjoint from all the spheres S_i and S'_j

and all 1-handles and 2-handles of X. Replace the 0-handle by its union with these arcs.

We note that the 0-handle now intersects all the spheres S_i and S'_j and contains certain intersection points of these with other spheres. However there may be other points of intersection of the sphere with other spheres and with the cores of the 2-handles. To each such intersection point we associate a 1-handle. Namely, if the sphere S intersects the sphere S' in the point p, we pick arcs in S and S' joining the 0-handle to p. The neighbourhood of the union of these two arcs gives a 1-handle. We can make a similar construction for an intersection of S with the core of a 2-handle.

After adding 1-handles as above, the spheres S_i and S'_j and each 2-handle in the handle-decomposition of X can be regarded as 2-handles attached to the union of the 0-handles and the 1-handles. Further, the normal subgroup generated by the boundaries of the 2-handles is the free group generated by the 1-handles in X. In particular, the words they represent in the fundamental group of the 1-skeleton do not involve the new 1-handles corresponding to the intersection points.

We now extend this to a handle-decomposition of all of W. As W is simply-connected, the normal subroup generated by the boundaries of the 2-handles in the 1-skeleton is the fundamental group of the 1-skeleton. Hence after adding cancelling 2-handle and 3-handle pairs, by Tietze's theorem we can perform handle slides to that corresponding to each one handle there is a 2-handle whose attaching circle cancels it in the fundamental group.

We take B to be the union of the 1-handles, the corresponding cancelling 2-handles and the 2-handles corresponding to the spheres S_i, the spheres S'_j and the 2-handles in the handle-decomposition of X. Then we claim that B is as required.

Observe that the union of the 0-handle, 1-handle and the cancelling 2-handles is contractible by construction. The two handles associated to the spheres S_i and S'_i have trivial boundary and hence represent homology classes. By construction, these homology classes form the basis of $k\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ in the above decomposition. Further, the 1-handles and 2-handles obtained from the handle decomposition of X give the homology of X. Thus B is as claimed.

□

We can now complete the proof of Theorem 1.3. Recall that the spheres S_i are all contained in B and surgery about these spheres gives M. Thus performing surgery on B about these spheres gives a subset $A \subset M$, with the closure of $M - A$ diffeomorphic to the closure of $W - B$.

Similarly, we perform surgery about the spheres S'_i in B to get a subset of $N\#X$ that contains X. By collapsing X we get a subset $C \subset X$. As before the closure of $N - C$ is diffeomorphic to the closure of $W - B$, hence is diffeomorphic to the closure of $M - A$.

By using the methods of Whitehead, f is homotopic to a map $g : M \to N$ that restricts to the above diffeomorphism from $M - A$ to $N - C$. Further, as B is simply connected and $H^2(W) = H^2(N) \oplus H^2(B)$, C is contractible. Thus, g and C are as required. □

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